MIRROR SYMMETRY IN TWO STEPS: A–I–B

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ABSTRACT. We suggest an interpretation of mirror symmetry for toric varieties via an equivalence of two conformal field theories. The first theory is the twisted sigma model of a toric variety in the infinite volume limit (the A–model). The second theory is an intermediate model, which we call the I–model. The equivalence between the A–model and the I–model is achieved by realizing the former as a deformation of a linear sigma model with a complex torus as the target and then applying to it a version of the T–duality. On the other hand, the I–model is closely related to the twisted Landau-Ginzburg model (the B–model) that is mirror dual to the A–model. Thus, the mirror symmetry is realized in two steps, via the I–model. In particular, we obtain a natural interpretation of the superpotential of the Landau-Ginzburg model as the sum of terms corresponding to the components of a divisor in the toric variety. We also relate the cohomology of the supercharges of the I–model to the chiral de Rham complex and the quantum cohomology of the underlying toric variety.

INTRODUCTION

Two-dimensional supersymmetric sigma models have attracted a lot of attention in recent years. These models are rich enough to display many important and non-trivial physical phenomena, understanding which may help us gain insights into more difficult models, such as the four-dimensional gauge theories. One of the most interesting phenomena is mirror symmetry which is a duality between a type A twisted sigma model and a type B twisted topological theory, such as a Landau-Ginzburg model (see, e.g., [18]). The advent of mirror symmetry has led to spectacular conjectures and results in mathematics, bringing together such diverse topics as enumerative algebraic geometry, Gromov-Witten invariants, Floer cohomology, soliton equations and singularity theory (see the book [19] and references therein).

In this paper we suggest an interpretation of mirror symmetry for toric varieties. We show that there is a certain conformal field theory (the "I–model") that is intermediate between the type A twisted sigma model and the type B twisted Landau-Ginzburg model. On the one hand, this model is equivalent to the sigma model of a toric variety in the infinite volume limit, considered as a conformal field theory, and on the other hand its BPS sector is closely related to the BPS sector of the corresponding Landau-Ginzburg model. Let us describe this correspondence in more detail.

Sigma model in the infinite volume. Consider the type A twisted $N = (2,2)$ supersymmetric sigma model with a target Kähler manifold M. This model is believed to define a superconformal quantum field theory if the Kähler metric is Ricci flat, i.e., if M is a Calabi-Yau manifold. However, we will argue in this paper that a suitable *infinite volume* limit of the twisted sigma model defines a conformal field theory for more general target manifolds.

This infinite volume limit is defined at the classical level by passing to a suitable "first order formalism" Lagrangian, which has previously been considered in the literature in [29, 3, 6] and more recently in [2, 20]. Rescaling the Kähler metric by a parameter t, we find that the first order Lagrangian has a well-defined limit even as $t \to \infty$. In this limit we obtain a conformally invariant Lagrangian, which describes what is natural to call the infinite volume limit of the twisted sigma model (see Section 1.1 for details).

Quantization of a first order Lagrangian could be non-trivial and even problematic in some cases. However, in the twisted $N = (2,2)$ supersymmeric theory that we are considering it is expected that all potential anomalies cancel out and the theory remains conformally invariant at the quantum level as well. Moreover, the corresponding path integral over all maps $\Phi : \Sigma \to M$, where Σ is a Riemann surface (the worldsheet), has a nice geometric interpretation as the delta-form supported on the subspace of *holomorphic maps* $\Phi : \Sigma \to M$. When we deform the Lagrangian back to the finite volume, i.e., to finite values of t, we obtain what looks like a "smoothening" of this delta-form, or, more precisely, the Mathai-Quillen representative of the Euler class of an appropriate vector bundle over the space of maps, see [6]. Hence it is natural to think that in the infinite volume limit the path integral localizes on the holomorphic maps, i.e., it can be represented as a sum of integrals over the finite-dimensional moduli spaces of holomorphic maps of different degrees (see Section 1.2). This is what one expects in the type A twisted sigma model in the infinite volume limit as explained by E. Witten in [30, 31].

We wish to view the model in the infinite volume first and foremost as a topological conformal field theory. In particular, it should come with a Hilbert space combining chiral and anti-chiral states, and a state-field correspondence. Correlation functions should be defined for any Riemann surface Σ with marked points $x_1, \ldots, x_n$ (and possibly germs of local coordinates at those points), and a collection of local operators inserted at those points. These correlation functions may be viewed as differential forms on the moduli space $\mathcal{M}_{g,n}$ of pointed curves $(\Sigma, (x_i))$ (see Section 1.3).

Part of this structure is captured by the *Gromov-Witten invariants*, which appear as integrals of the differential forms corresponding to particular observables over a compactification of $\mathcal{M}_{g,n}$ (see Section 1.3 for more details).

Another ingredient of this conformal field theory is a sheaf of chiral algebras over M, called the *chiral de Rham complex*, introduced in [26]. It is defined by gluing free chiral algebras on the overlaps of open subsets of M isomorphic to $\mathbb{C}^n$. From the point of view of the twisted sigma model, this chiral algebra corresponds to the cohomology of the right moving supercharge in the *perturbative* regime, i.e., without counting the instanton contributions, as explained in [34, 20]. In order to understand the correlation functions of the sigma model and in particular to include the instanton corrections, it is

necessary to go beyond the chiral algebra and consider the full conformal field theory. This is one of the goals of the present paper.

Non-linear sigma models as deformations of free field theories. There is one case when the sigma model can certainly be defined as a conformal field theory, and this is the case of the target manifolds with a flat metric, such as a flat space $\mathbb{C}^n$ or a torus (for a detailed treatment of the latter, see [21]). We will consider in Section 2 the intermediate case of the sigma model in the infinite volume with the target manifold a complex torus $(\mathbb{C}^\times)^n$, which we call the *toric sigma model*. This is a free conformal field theory, but we will show that it exhibits some non-trivial effects, such as the appearance of holomorphic analogues of vortex operators, which we call *holomortex* operators.

We will then define in Sections 3 and 4 the conformal field theory governing a non-linear sigma model of a toric variety in the infinite volume as a *deformation*, in the sense of A. Zamolodchikov [35], of the toric sigma model, by some explicitly written exactly marginal operators. By its very definition, this deformed conformal field theory will include the instanton effects corresponding to holomorphic maps of non-zero degree.

To illustrate our main idea, it is instructive to look at the case of the sigma model with the target $\mathbb{P}^1$ in the infinite volume limit, obtained by quantization of the corresponding first order Lagrangian. We wish to obtain it as a deformation of the toric sigma model with the target $\mathbb{C}^\times$ which we realize as the quotient $\mathbb{C}/2\pi i\mathbb{Z}$. This is a free conformal field theory with the basic chiral fields $X(z), p(z),\ \psi(z),\ \pi(z)$, and their anti-chiral partners with the action

$$(0.1) \qquad \frac{i}{2\pi} \int_\Sigma d^2z\ \left(p\partial_{\bar z}X + \overline{p}\partial_z\overline{X} + \pi\partial_{\bar z}\psi + \overline{\pi}\partial_z\overline{\psi} \right).$$

The field $X(z)$ corresponds to a linear coordinate on $\mathbb{C}/2\pi i\mathbb{Z}$, and so is defined modulo $2\pi i\mathbb{Z}$.

As discussed above, the correlation functions of this model are given by integrals over the space of holomorphic maps $\Sigma \to \mathbb{C}^\times$. For compact Σ, all such maps are necessarily constant. Therefore the correlation functions reduce to integrals over the zero mode (i.e., over the image of the constant map $\Phi : \Sigma \to \mathbb{C}^\times$), as expected in a free field theory.

How can we interpret holomorphic maps $\Sigma \to \mathbb{P}^1$ within the framework of this free field theory? Such maps may be viewed as holomorphic maps $\Sigma \backslash \{w_i^\pm\} \to \mathbb{C}/2\pi i\mathbb{Z}$ with logarithmic singularities at some points $w_1^\pm, \ldots, w_N^\pm$, where this map behaves as $\pm \log(z - w_i^\pm)$. These singular points correspond to zeroes and poles of $\exp\Phi$, and generically they will be distinct. Our proposal is that *we can create these singularities of Φ by inserting in the correlation function of the linear sigma model certain vertex operators $\Psi_\pm(w_i^\pm)$.*

The defining property of the operators $\Psi_\pm(w)$ (up to a scalar) is that their operator product expansion (OPE) with $X(z)$ should read

$$(0.2) \qquad X(z)\Psi_\pm(w) = \pm \log(z - w)\Psi_\pm(w).$$

Given such operators, we can write a given function (in the case of Σ of genus zero)

$$\Phi(z) = c + \sum_{i=1}^{n} \log(z - w_i^+) - \sum_{i=1}^{n} \log(z - w_i^-)$$

as the correlator

$$\Phi(z) = \langle X(z) \prod_{i=1}^{n} \Psi_+(w_i^+) \prod_{i=1}^{n} \Psi_-(w_i^-) \delta^2(X(\infty) - c) \psi(\infty) \overline{\psi}(\infty) \rangle$$

(the term involving the delta-function and the fermions will give, upon the integration over the zero modes of X and ψ, the normalization condition $\Phi(\infty) = c$). Thus, we can create all instantons of the $\mathbb{P}^1$ sigma model, that is holomorphic maps $\Sigma \to \mathbb{P}^1$, as correlation functions in the toric sigma model of the above form (the case of Σ of genus greater than zero will be discussed in Section 3.1).

The property (0.2) is satisfied by the following fields

$$\Psi_\pm(w, \overline{w}) = \exp\left(\mp i \int_{w_0}^{w} (p(z)dz + \overline{p}(\overline{z})d\overline{z}) \right),$$

which are examples of the holomortex operators mentioned above. Including these operators in the correlation functions and allowing the points $w_i^\pm$ to vary over Σ is equivalent to deforming the action (0.1) with the term

$$q^{1/2} \int_\Sigma \left(\Psi_+^{(2)} + \Psi_-^{(2)} \right),$$

where $\Psi_\pm^{(2)}$ are the cohomological descendants

$$\Psi_\pm^{(2)} = \Psi_\pm(w, \overline{w}) \pi(w) \overline{\pi}(\overline{w}) dw d\overline{w}.$$

The resulting deformed theory appears to be equivalent to the sigma model with the target $\mathbb{P}^1$ in the infinite volume limit (in the sense explained in Section 3). By construction, the part of a correlation function of this deformed theory that corresponds to degree n maps $\Sigma \to \mathbb{P}^1$ will appear with the overall factor q^n.

More generally, suppose that we are given a smooth compact Kähler manifold M with an open dense submanifold M_0 with a linear structure. The complement $C = M \backslash M_0$ is a compactification divisor, which is a union of irreducible components $C_1, \ldots, C_N$. The linear sigma model corresponding to M_0 is a free superconformal field theory, and we wish to describe the non-linear model with the target M in terms of this theory. Let us observe that a generic holomorphic map $\Phi : \Sigma \to M$ will take values in C at a finite set of points $x_1, \ldots, x_n$, and generically we will have $\Phi(x_j) \in C_{k_j}$ and $\Phi(x_j) \notin C_l, l \neq k_j$. To account for such maps we need to insert some vertex operators Ψ_{k_j} corresponding to the compactification divisors C_{k_j} at the points $x_j, j = 1, \ldots, n$. It is then natural to expect that the non-linear sigma model with the target M in the infinite volume limit can be described as the deformation of the free field theory corresponding to the target M_0 by means of the operators $\Psi_k^{(2)}, k = 1, \ldots, N$, where $\Psi_k^{(2)}$ is the $(1,1)$–form counterpart of Ψ_k obtained via the cohomological descent. To solve the theory we therefore need to identify explicitly the suitable vertex operators $\Psi_k, k = 1, \ldots, n$, corresponding to the

compactification divisors. In general, they may be highly non-local and given by very complicated formulas.

While finding these operators may seem like a daunting task in general, it turns out that in the case when the target is a toric variety, they can be written down quite explicitly. Such a variety M comes with a natural open dense subset M_0 isomorphic to $(\mathbb{C}^\times)^n$ and the compactification divisors are naturally parameterized by the one-dimensional cones in the fan defining M. We construct explicitly the vertex operators corresponding to these compactification divisors in Section 4. These operators may be viewed as holomorphic counterparts of the vortex operators familiar from the free bosonic theory compactified on a torus. We will argue that the deformation of the action by these operators changes the topology of the target manifold and deforms a free field theory to a non-linear sigma model with the target M.

As in the case of $\mathbb{P}^1$, we expect that the sigma model with the target M, which is a smooth compact Fano toric variety, is equivalent to a deformation of the free field theory with the target $(\mathbb{C}^\times)^n$ by the holomortex operators corresponding to the irreducible components of the compactification divisor.

As a consistency check, we compute in Section 5.4 the cohomology of the right moving supercharge in our deformed theory, making a connection to the results of L. Borisov [4] and F. Malikov–V. Schechtman [25]. In particular, we show that in the case of $M = \mathbb{P}^n$ this cohomology is equal to the quantum cohomology of $\mathbb{P}^n$. On the other hand, in a certain limit we obtain the cohomology of the chiral de Rham complex of M. This is consistent with the assertion of [34, 20] that the chiral de Rham complex should appear as the cohomology of the right moving supercharge of the type A twisted sigma model in the perturbative regime.

I–model and mirror symmetry. Next, we consider the question as to what is the meaning of *mirror symmetry* from the point of view of our description of the sigma model of a toric variety as a deformation of a free field theory. The first step in answering this question is to perform a kind of T–duality transform of the free field theory with the target $(\mathbb{C}^\times)^n$.

In the case of $\mathbb{P}^1$, before the deformation, we have the free field theory with the target $\mathbb{C}^\times$. The dual of this theory turns out to be the ordinary sigma model with the target being the cylinder $\mathbb{R} \times S^1$ equipped with the metric of *Minkowski* signature. Let R and U be the coordinates on $\mathbb{R}$ and $S^1 = \mathbb{R}/2\pi$, respectively. Under the T–duality the local fields p and X become more complicated, but the complicated fields, like the holomortex operators, become simple. In fact, we have the following transformation:

$$p\,dz + \overline{p}\,d\overline{z} = dU,$$

and so the holomortex operators $\Psi_\pm$ turn out to be simply the exponential fields $e^{\mp iU}$. The field R coincides with the field $\frac{1}{2}(X + \overline{X})$ of the original theory. Therefore e^R coincides with the field $|e^X|$, the absolute value of the holomorphic coordinate on $\mathbb{P}^1$ compactifying the target $\mathbb{C}^\times$. The action of the deformed dual theory reads

$$(0.3) \quad \frac{i}{2\pi} \int_\Sigma d^2z \, \left(\partial_z U \partial_{\overline{z}} R + \partial_{\overline{z}} U \partial_z R + \pi \partial_{\overline{z}} \psi + \overline{\pi} \partial_z \overline{\psi} \right) + q^{1/2} \int_\Sigma (e^{iU} + e^{-iU}) \pi \overline{\pi} d^2z.$$

Thus, the correlation functions of the observables of this theory that depend only on the field R realize the corresponding correlation functions of the twisted sigma model, namely, those that depend only on $|e^X|$. But while the correlation functions of the twisted sigma model appear as sums over the instanton contributions, the dual description gives us their *non-perturbative* realization!

Let us compare the action (0.3) to the action of the Landau-Ginzburg model with the target $\mathbb{C}$ and the Landau-Ginzburg superpotential

$$W = q^{1/2}(e^{iY} + e^{-iY}),$$

where Y is a chiral superfield:

$$(0.4) \quad \frac{1}{2\pi} \int_\Sigma d^2z \ \left(\partial_z\varphi\partial_{\overline{z}}\overline{\varphi} + \partial_{\overline{z}}\varphi\partial_z\overline{\varphi} + i\chi_+\partial_{\overline{z}}\overline{\chi}_+ + i\chi_-\partial_z\overline{\chi}_-\right)$$

$$+ q^{1/2}\int_\Sigma (e^{i\varphi} + e^{-i\varphi})\chi_+\chi_- d^2z.$$

We observe that the two actions look similar: if we "analytically continue" the theory with the action (0.3), allowing the fields U and R to become complex-valued fields φ and $\overline{\varphi}$, which are complex conjugate to each other, and rename the fermions as follows:

$$\pi \mapsto \chi_-, \quad \overline{\pi} \mapsto \chi_+, \quad \psi \mapsto \overline{\chi}_-, \quad \overline{\psi} \mapsto \overline{\chi}_+,$$

then the action (0.3) becomes the action (0.4). This means that the correlation functions in the two theories should be related by a kind of analytic continuation. However, we wish to stress the models with the actions (0.3) and (0.4) are *different*. For example, in the model (0.3) the field U is real periodic, and R is real non-periodic, while in the model (0.4) the fields φ and $\overline{\varphi}$ are complex (conjugate to each other) and both periodic.

It is instructive to compare the supersymmetry charges in the above models. For simplicity we consider the case when $q = 0$. In the original A–model with the action (0.1) the supercharge is

$$\int (\psi p dz + \overline{\psi}\,\overline{p}d\overline{z}).$$

This is a de Rham type supercharge, because under its action $X \mapsto \psi, \overline{X} \mapsto \overline{\psi}$. In the T–dual theory with the action (0.3) (with $q = 0$) the supercharge becomes

$$\int (\psi\partial_z U dz + \overline{\psi}\partial_{\overline{z}}U d\overline{z}).$$

Under the "analytic continuation" that we discussed above, it becomes the supercharge of the type B twisted Landau-Ginzburg model with the action (0.4) (with $q = 0$):

$$\int (\overline{\chi}_-\partial_z\varphi dz + \overline{\chi}_+\partial_{\overline{z}}\varphi d\overline{z}).$$

This is now a Dolbeault type supercharge, because under its action $\varphi \mapsto 0, \overline{\varphi} \mapsto \overline{\chi}_- + \overline{\chi}_+$. Thus, the T–duality indeed transforms a de Rham type supercharge of the A-model to a Dolbeault type supercharge of the B–model, as expected in mirror symmetry. Note that the interpretation of the fermionic fields is very different in the two theories, and this underscores the highly non-local nature of the mirror symmetry.

Traditionally the Landau-Ginzburg model is defined by adding to the action of the supersymmetric linear sigma model the term $\int d^2z d^2\theta W(Y) + \int d^2z d^2\overline{\theta}\, \overline{W}(\overline{Y})$. Usually, one chooses $\overline{W}(\overline{Y})$ to be complex conjugate of $W(Y)$. But in a type B twisted Landau-Ginzburg model there is an essential difference between the first and the second terms: while the integrand in the first one is a $(1,1)$–form, the integrand in the second is a $(0,0)$–form, and hence to integrate it one needs to pick a metric on the worldsheet. This breaks conformal invariance. That is why in the action (0.4) we have set $\overline{W} = 0$, for otherwise the theory would not be conformally invariant.

The Landau-Ginzburg model with the (twisted) superpotential W, where W is as above, and its complex conjugate $\overline{W}$ has been considered by K. Hori and C. Vafa [18] (see also [13, 5, 7, 16]). They showed that its correlation functions *in the BPS sector* are related to those of the twisted sigma model of $\mathbb{P}^1$, which is the sense in which the two theories are mirror dual to each other. Note that $\overline{W}$ is Q–exact, and the possibility of setting $\overline{W}$ to 0 was mentioned in [22] and [18], Sect. 6.

The point of our construction is that in addition to the twisted sigma model and the Landau-Ginzburg model, which are usually considered in the study of mirror symmetry, there is an intermediate model, or the "I–model", described by the action (0.3). This is a conformal field theory that has two properties: on the one hand it should be equivalent to the type A twisted sigma model with the target $\mathbb{P}^1$ in the infinite volume, which is also a conformal field theory. In other words, *all* correlation functions in the two models are equivalent, not just in the BPS sector. On the other hand, the BPS sector of the I–model is closely related to the BPS sector of the type B twisted Landau-Ginzburg model considered in [18] (see the discussion in Section 3.2 for more details).

This conclusion leads to a curious observation that the correlation functions of the field e^R in the I–model (which corresponds to $e^{\overline{\varphi}}$ in the Landau-Ginzburg model (0.4)) encode the correlation functions of the field $|e^X|$ of the sigma model with the target $\mathbb{P}^1$. Thus, one can actually *see the $\mathbb{P}^1$ instantons*, and not just the correlation functions of the BPS states, in the framework of the I–model (or the Landau-Ginzburg model)!

We define a similar I–model for an arbitrary toric variety. Then the corresponding deformation term in the Lagrangian is equal to the sum $\sum_{k=1}^N e^{-iU_k}\pi_{(k)}\overline{\pi}_{(k)}$ over the components of the compactification divisor of our toric variety. The fields U_k satisfy constraints reflecting the structure of the fan defining the toric variety M. For example, in the case when $M = \mathbb{P}^n$ we have $N = n+1$, and the fields U_k satisfy the familiar constraint $\prod_{k=1}^{n+1} e^{-iU_k} = q$. Thus, we immediately recognize that, after the analytic continuation, we obtain a term that looks like the Landau-Ginzburg superpotential corresponding to $\mathbb{P}^n$ considered in [18]. We note that these superpotentials and the corresponding oscillating integrals representing correlation functions of the Landau-Ginzburg model had previously appeared in the mathematical work of A. Givental [16] on mirror symmetry.

We stress that in our approach the superpotential is generated because of our description of the sigma model with the target M (in the infinite volume limit) as a deformation of a free field theory, to which we apply the T–duality transform. Therefore the superpotential has a transparent geometric meaning. Namely, the summands appearing in the superpotential naturally correspond to the irreducible components

of the compactification divisor in M. The mirror symmetry can now be viewed as a corollary of the equivalence of the I–model and the A–model (sigma model with the target M in the infinite volume limit), as conformal field theories. We hope that the I–model will help us understand more fully the phenomenon of mirror symmetry.[1]

In the case of $\mathbb{P}^n$, the action of the I–model is very similar to the action of the $A_{n-1}^{(1)}$ affine Toda field theory, considered as a deformation of a free field theory. However, since the I–model is conformally invariant, its structure is actually more reminiscent of that of the conformal A_{n-1} Toda field theory. We can use the methods familiar from the Toda theory to determine the structure of the chiral sector of the I–model. We recall that in the case of an A_{n-1} Toda field theory the chiral algebra of integrals of motion is the $\mathcal{W}_n$–algebra [8, 12]. It appears as the subalgebra of those operators of the free field theory which commute with the *screening operators*, which are the residues of the operators deforming the action. Likewise, the $\mathcal{W}$–algebra in the I–model associated to a toric variety M consists of the operators that commute with the operators $\int e^{-iU_k}\pi_{(k)}dz, k = 1, \ldots, n+1$ (which can therefore be viewed as supersymmetric analogues of the screening operators), and it is possible to determine it explicitly. In doing so, we make a connection to the results of [4] (see also [9, 17, 25]) and show that this $\mathcal{W}$–algebra is isomorphic to the algebra of global sections of the chiral de Rham complex on M.

In a follow-up paper we will generalize our results to hypersurfaces in toric varieties, and, more generally, to complete intersections in toric varieties. This way we hope to obtain a realization of mirror symmetry for such varieties as an equivalence of conformal field theories in the sense explained above.

In a future work we plan to consider an analogue of this construction for the $(0, 2)$ supersymmetric sigma models. We believe that in the case when M is a flag manifold of a simple Lie group, this theory, when coupled to gauge theory, is closely related to the geometric Langlands correspondence. We also plan to apply similar methods to the study of four-dimensional supersymmetric Yang-Mills theories.

The paper is organized as follows. In Section 1 we discuss the sigma model in the infinite volume limit, at both classical and quantum levels. We explain how the first order Lagrangian (with a B–field term) arises in the infinite volume limit and the interpretation of the corresponding path integrals as integrals of differential forms on the moduli spaces of holomorphic maps. We then outline our idea of constructing non-linear sigma models as deformations of linear ones. We illustrate this idea on the example of the deformation of the target manifold from $\mathbb{C}$ to $\mathbb{P}^1$. In Section 2 we introduce the toric sigma model, which is the linear sigma model with the target $\mathbb{C}^\times$ in the infinite volume. We define the holomortex operators and the T–duality transform. We show that the T–dual model of the toric sigma model is the ordinary sigma model with the target being the cylinder equipped with a metric of Minkowski signature. In Section 3 we consider a deformation of the toric sigma model to the sigma model with the target $\mathbb{P}^1$. We then define the T–dual theory, which is our I–model. We give a sample computation of the correlation functions in the I–model and obtain explicit

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[1]It is instructive to compare our derivation of mirror symmetry to A. Polyakov's model of confinement in three dimensions [27].

formulas for the supercharges. We generalize these results to the case of an arbitrary compact smooth toric variety in Section 4. Finally, we discuss the operator formalism of these theories in Section 5, as well as their $\mathcal{W}$–algebras and the cohomologies of the supersymmetry charges.

1. Supersymmetric sigma model in the infinite volume limit

1.1. **Lagrangian description.** We start by describing the A twisted $N = (2,2)$ supersymmetric sigma model in the formalism of the first order, following [29, 3, 6]. Let Σ be a complex Riemann surface (worldsheet). We denote by z and $\overline{z}$ the local holomorphic and anti-holomorphic coordinates on Σ, and by $d^2z = idz \wedge d\overline{z}$ the corresponding integration measure on Σ. Let M be a complex Kähler manifold (target) with a fixed Kähler metric $g_{a\overline{b}}$.

We will denote by $X^a, a = 1, \ldots, N = \dim M$, local holomorphic coordinates on M, and by $X^{\overline{a}} = \overline{X^a}$ their complex conjugates. Given a map $\Phi : \Sigma \to M$, we consider the pull-backs of X^a and $X^{\overline{a}}$ as functions on Σ, denoted by the same symbols. We also have fermionic fields ψ^a and $\psi^{\overline{a}}, a = 1, \ldots, N$, which are sections of $\Phi^*(T^{1,0}M)$ and $\Phi^*(T^{0,1}M)$, respectively. The Levi-Civita connection on TM corresponding to the metric $g_{a\overline{b}}$ induces a connection on $\Phi^*(TM)$. The corresponding covariant derivatives have the form

$$D_{\overline{z}}\psi^a = \partial_{\overline{z}}\psi^a + \partial_{\overline{z}}X^b \cdot \Gamma^a_{bc}\psi^c,$$
$$D_z\psi^{\overline{a}} = \partial_z\psi^{\overline{a}} + \partial_z X^{\overline{b}} \cdot \Gamma^{\overline{a}}_{\overline{b}\overline{c}}\psi^{\overline{c}},$$

where $\Gamma^a_{bc} = g^{a\overline{b}}\partial_b g_{c\overline{b}}$.

Next, we introduce auxiliary fields p_a which will play the role of the "Lagrange multipliers" corresponding to the equations $\partial_{\overline{z}}X^a = 0$, and their complex conjugates $p_{\overline{a}}$. Their fermionic super-partners will be denoted by π_a and $\pi_{\overline{a}}$. These are sections of $\Phi^*(\Omega^{1,0}M) \otimes \Omega^{1,0}\Sigma$ and $\Phi^*(\Omega^{0,1}M) \otimes \Omega^{0,1}\Sigma$, respectively.

We write down the action for these fields following [29] (formula (2.14)):

$$(1.1) \quad I_t = \frac{1}{2\pi}\int_\Sigma d^2z \ \left(ip_a\partial_{\overline{z}}X^a + ip_{\overline{a}}\partial_z X^{\overline{a}} + i\pi_a D_{\overline{z}}\psi^a + i\pi_{\overline{a}}D_z\psi^{\overline{a}}\right.$$
$$\left. - t^{-1}R^{a\overline{b}}{}_{c\overline{d}}\pi_a\pi_{\overline{b}}\psi^c\psi^{\overline{d}} + t^{-1}g^{a\overline{b}}p_a p_{\overline{b}}\right),$$

where t is a parameter (the "radius"). The equations of motion for $p_a, p_{\overline{a}}$ are as follows:

$$(1.2) \qquad\qquad p_a = -itg_{a\overline{b}}\partial_z X^{\overline{b}},$$

$$p_{\overline{a}} = -itg_{\overline{a}b}\partial_{\overline{z}} X^b.$$

Remark 1.1. Formulas (1.2) seem to indicate that the complex conjugate of p^a is equal to $-p^{\overline{a}}$, which is misleading. In fact, the substitution (1.2) is formal and only makes sense under the path integral. It corresponds to completing the action to a square and integrating out the variables p^a and $p^{\overline{a}}$. $\qquad\qquad\square$

Substituting these formulas back into (1.1), we obtain the action

$$\widetilde{I}_t = \frac{1}{2\pi} \int_\Sigma d^2z \, \left(tg_{a\overline{b}}\partial_{\overline{z}}X^a \partial_z X^{\overline{b}} + i\pi_a D_{\overline{z}}\psi^a + i\pi_{\overline{a}} D_z \psi^{\overline{a}} - t^{-1}R^{a\overline{b}}{}_{c\overline{d}}\pi_a\pi_{\overline{b}}\psi^c\psi^{\overline{d}} \right).$$

This is the action of the A–twisted $N = (2, 2)$ supersymmetric sigma model with the target M and the B–field $-\frac{t}{2\pi}\omega$, where $\omega = \frac{i}{2}g_{a\overline{b}}dX^a \wedge dX^{\overline{b}}$ is the Kähler form on M, introduced in [29, 31]. The corresponding metric on M is $tg_{a\overline{b}}$. Thus, the action (1.1) describes this model. In the infinite volume limit $t \to \infty$ the action (1.1) becomes

$$(1.3) \qquad I_\infty = \frac{i}{2\pi} \int_\Sigma d^2z \, \left(p_a\partial_{\overline{z}}X^a + p_{\overline{a}}\partial_z X^{\overline{a}} + \pi_a D_{\overline{z}}\psi^a + \pi_{\overline{a}}D_z\psi^{\overline{a}} \right).$$

This action is conformally invariant, and it has two supersymmetries: one is mapping

$$X^a \mapsto \psi^a, \qquad \psi^a \mapsto 0,$$

$$\pi_a \mapsto -p_a - \Gamma^b_{ac}\pi_b\psi^c,$$

$$p_a \mapsto \Gamma^b_{ac}p_b\psi^c,$$

and the other does the same to their complex conjugates.

1.2. **The path integral.** The action (1.3) describes a conformal field theory governing the infinite volume limit of the A–twisted sigma model. We wish to understand the corresponding quantum field theory.

The first observation is that the path integral $\int[Dp][D\pi]e^{-I_\infty}$, considered as a differential form on the space of maps $\Sigma \to M$, may be viewed as the integral representation of the delta-function differential form supported on the space of holomorphic maps $\Sigma \to M$.

To see this, consider a finite-dimensional model situation: a complex vector space $\mathbb{C}^M$ and functions $f^a, a = 1, \ldots, N$, defining a codimension N complex subvariety $C \subset \mathbb{C}^M$. Then the delta-like differential form supported on this subvariety has the following integral representation:

$$\delta_C = \int \prod_a dp_a dp_{\overline{a}}d\pi_a d\pi_{\overline{a}} \exp\left(-ip_a f^a - i\overline{p}_a\overline{f}_a - i\pi_a df_a - i\overline{\pi}_a d\overline{f}_a\right).$$

This delta-form may be viewed as the limit, when $t \to \infty$, of the regularized integral

$$\delta_{C,t} = \int \prod_a dp_a dp_{\overline{a}}d\pi_a d\pi_{\overline{a}} \exp\left(-ip_a f^a - i\overline{p}_a\overline{f}_a - i\pi_a df_a - i\overline{\pi}_a d\overline{f}_a - t^{-1}p_a p_{\overline{a}}\right).$$

Comparing these formulas to (1.1) and (1.3), we see that the path integral

$$(1.4) \qquad \int [Dp][D\pi] e^{-I_\infty}$$

looks like a delta-like form supported on the solutions of the equation $\partial_{\overline{z}} X^a = 0$, i.e., on the holomorphic maps, while $\int [Dp][D\pi] e^{-I_t}$ may be viewed as its regularized version. Alternatively, and more precisely, one may say that the integral $\int [Dp][D\pi] e^{-I_t}$ looks like the Mathai-Quillen representative of the Euler class of an appropriate vector bundle over the space of maps $\Sigma \to M$ (see [6], § 13.6).

Motivated by this analogy, it is natural to expect that in the infinite volume limit the correlation functions in our theory will correspond to sums of integrals of differential forms over different connected components of the moduli space of holomorphic maps $\Sigma \to M$, as explained in [30]. Particular examples of these functions give rise to the Gromov-Witten invariants of M [31].

The connected components of the moduli space of holomorphic maps $\Sigma \to M$ are labeled by $H_2(M)$. Choosing a basis in $H_2(M)$, we can label them by k–tuples of integers $(n_1, \ldots, n_k)$. It is customary to weight the contribution to the path integral corresponding to the component of the space of holomorphic maps $\Sigma \to M$ of degree $(n_1, \ldots, n_k)$ with the coefficient $q_1^{n_1} \ldots q_k^{n_k}$ (we choose this basis in such a way that non-zero contributions come from $n_i \geq 0$). This can be achieved by adding to the action I_∞ the topological term $\sum_i \frac{u_i}{2\pi} \int_\Sigma \Phi^*(\varpi^i)$. Here $\{\varpi^i\}$ is the basis of the Kähler cone of M that is dual to the above basis of $H_2(M)$ and the u_i's are the coupling constants such that $q_i = e^{-u_i}$.

The corresponding path integral is then the sum over $n_1, \ldots, n_k \geq 0$ of terms corresponding to the holomorphic maps $\Sigma \to M$ of degrees $(n_1, \ldots, n_k)$ with coefficients $q_1^{n_1} \ldots q_k^{n_k}$. This path integral may be obtained as the $t \to \infty$ limit of a sigma model path integral as follows. We simply add to the action the B–field $-\frac{t}{2\pi} \omega + \frac{1}{2\pi} \varpi$, where $\omega = \frac{i}{2} g_{a\overline{b}} dX^a \wedge dX^{\overline{b}}$ is the Kähler form on M and $\varpi = \sum_i u_i \varpi^i$. Then the bosonic part of the action will read

$$\frac{1}{2\pi} \int_\Sigma d^2 z \ \left(\frac{t}{2} \left(g_{a\overline{b}} \partial_{\overline{z}} X^a \partial_z X^{\overline{b}} + g_{a\overline{b}} \partial_z X^a \partial_{\overline{z}} X^{\overline{b}} \right) + \frac{t}{2} \left(g_{a\overline{b}} \partial_{\overline{z}} X^a \partial_z X^{\overline{b}} - g_{a\overline{b}} \partial_z X^a \partial_{\overline{z}} X^{\overline{b}} \right) \right.$$

$$\left. + \sum_i u_i \Phi^*(\varpi^i) \right) = \frac{1}{2\pi} \int_\Sigma d^2 z \ t g_{a\overline{b}} \partial_{\overline{z}} X^a \partial_z X^{\overline{b}} + \sum_i \frac{u_i}{2\pi} \int_\Sigma \Phi^*(\varpi^i).$$

In terms of the first order variables this becomes

$$\frac{1}{2\pi} \int_\Sigma d^2 z \ \left(i p_a \partial_{\overline{z}} X^a + i p_{\overline{a}} \partial_z X^{\overline{a}} + t^{-1} g^{a\overline{b}} p_a p_{\overline{b}} \right) + \sum_i \frac{u_i}{2\pi} \int_\Sigma \Phi^*(\varpi^i).$$

Therefore in the limit $t \to \infty$ the path integral will indeed give us the desired sum over $n_1, \ldots, n_k \geq 0$ weighted with coefficients $q_1^{n_1} \ldots q_k^{n_k}$, where $q_i = e^{-u_i}$.

Proper definition of the path integral (1.4) for worldsheets Σ of genus greater than zero requires a prescription for the integration of the zero modes of the fields p_a and $p_{\overline{a}}$. [2] The most evident possibility to do so is to add the term of the form $\epsilon G^{a\overline{b}} p_a p_{\overline{b}}$ to

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[2]We thank N. Nekrasov for a discussion of this point.

the action and consider the limit $\epsilon \to 0$. However, if we choose $G^{a\overline{b}}$ to be the inverse of a Kähler form on M, this will bring us back to the finite volume and spoil conformal invariance if M is not Calabi-Yau. But we can take $G^{a\overline{b}}$ to be any tensor in $T^{1,1}M$ of the following form. Suppose that we have a flat Kähler metric on an open dense subset M_0 of M, such that its inverse is a section of $T^{1,1}M_0$ that extends to a section on the entire M. We can then take this extension as our $G^{a\overline{b}}$. Then we can regularize the integrals over the zero modes of the p_a's and $p_{\overline{a}}$'s without violating conformal invariance of the theory. Such tensors can be easily constructed for Fano toric varieties, and we will see examples of that below. We also remark that for general Fano manifolds the zero modes disappear altogether when the genus of Σ is fixed and the degree of the map $\Sigma \to M$ is sufficiently high.

Remark 1.2. The action (1.3) is conformally invariant, and we expect that the corresponding quantum field theory is also conformally invariant, for any Kähler manifold M. However, in the case of non-Ricci flat Kahler manifolds non-zero β–function is developed and the theory becomes non-conformal for finite values of t, even though the deformation to finite volume is achieved by adding the operator $V = \sum_{a,\overline{b}} g^{a\overline{b}} p_a p_{\overline{b}}$ of dimension (1,1). In general, consider the basis V_a in the space of operators of dimension (1,1). The $|z - w|^{-2}$ term in their operator product expansion reads as follows

$$V_a(z)V_b(w) \sim \frac{C_{ab}^c V_c(w)}{|z - w|^2}.$$

Then the theory with interaction $t^a V_a$ has the beta-function equal to $t^a t^b C_{ab}^c V_c$. In our case, the OPE of the above operator V with itself contains $|z-w|^{-2}$ with the coefficient proportional to $R^{a\overline{b}} p_a p_{\overline{b}}$, where $R_{a\overline{b}}$ is the Ricci curvature of M [23]. Therefore, if M is not Calabi-Yau, the sigma model in the finite volume is not conformally invariant. However, in the infinite volume limit the beta-function vanishes and the theory becomes conformally invariant, even for manifolds that are not Calabi-Yau. $\qquad\square$

1.3. **Correlation functions.** Correlation functions in our model are defined for any Riemann surface Σ with marked points $x_1, \ldots, x_n$, and a collection of local operators inserted at those points. In a general conformal field theory with central charge $c = 0$ correlation functions are functions on the moduli space $\mathcal{M}_{g,n}$ of pointed curves $(\Sigma, (x_i))$.[3] But our theory carries a supersymmetry charge Q such that the stress tensor $T(z)$ is Q–exact: $T(z) = [Q, G(z)]_+$, and similarly for the anti-chiral fields, and so it has the structure of *topological conformal field theory*. In a topological conformal field theory we can construct not only functions, but also *differential forms* on the moduli space $\mathcal{M}_{g,n}$, by inserting integrals of the fields $G(z)$ and $\overline{G}(\overline{z})$ (see [33, 36]). Let us recall this construction.

Suppose for simplicity that $n > 0$, and let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be some local operators inserted at the points $x_1, \ldots, x_n$. We will explain how to construct holomorphic differential forms. The construction is easily generalized to arbitrary forms. We note that the

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[3] we may also need to choose non-zero tangent vectors, or even germs of local coordinates, at the marked points, but in the discussion below we will omit them

holomorphic tangent space to the moduli space $\mathcal{M}_{g,n}$ at $(\Sigma, (x_i))$ is isomorphic to the double quotient

$$\Gamma(\Sigma\backslash\{x_1, \ldots, x_n\}, T^{1,0}\Sigma)\backslash \bigoplus_{i=1}^{n} \mathbb{C}((t_i))\partial_{t_i} / \bigoplus_{i=1}^{n} \mathbb{C}[[t_i]]\partial_{t_i},$$

where $T^{1,0}\Sigma$ is the holomorphic tangent bundle of Σ (see, e.g., [14], Sect. 17.3, and references therein). Now any holomorphic vector field on the punctured disc near x_i, $\xi_i = f_i(t_i)\partial_{t_i} \in \mathbb{C}((t_i))\partial_{t_i}$ defines a tangent vector in $T^{1,0}_{(\Sigma,(x_i))}\mathcal{M}_{g,n}$. To define a differential $(k,0)$–form on $\mathcal{M}_{g,n}$ corresponding to $\mathcal{O}_1, \ldots, \mathcal{O}_n$ we need to describe its values on k–tuples of holomorphic tangent vectors of the above form. Let us suppose that we have tangent vectors corresponding to the vector fields $\xi_j^{(1)}, \ldots, \xi_j^{(\alpha_j)}$ at the point x_j. Then, by definition, the value of this $(k,0)$–form on these tangent vectors is just the correlation function

$$\left\langle \prod_{j=1}^{n} \int \xi_j^{(1)} G(z_j^{(1)}) \ldots \int \xi_j^{(\alpha_j)} G(z_j^{(\alpha_j)}) \mathcal{O}_j \right\rangle.$$

In other words, we "dress" the local operator inserted at x_j by contour integrals of $G(z)$ coupled to the vector fields $\xi_j^{(1)}, \ldots, \xi_j^{(\alpha_j)}$. To obtain more general differential forms, we should use the anti-chiral field $\overline{G}(\overline{z})$ as well.

If the observables $\mathcal{O}_j$ have definite fermionic charges, then among all of these differential forms there is at most one that is non-zero. Its degree is determined by the corresponding fermionic charge conservation law.

What do these differential forms look like? Typical observables of the theory are differential forms on M, and Q acts on them as the de Rham differential. Let $\mathcal{M}_{g,n}(M,\beta)$ be the moduli space of $(\Sigma, (x_i), \Phi)$, where Σ and (x_i) are as above and Φ is a holomorphic map $\Sigma \to M$ of degree β. Then we have a forgetful map $\mathcal{M}_{g,n}(M,\beta) \to \mathcal{M}_{g,n}$. Suppose we want to compute the correlation functions of the local operators corresponding to differential forms $\omega_i, i = 1, \ldots, n$ on M, not necessarily closed. Then we should take the cup product of the pull-backs of the ω_i's to $\mathcal{M}_{g,n}(M,\beta)$ under the evaluation maps, and take the push-forward of the resulting differential form to $\mathcal{M}_{g,n}$. If the ω_i's are smooth and have compact support, then one can show that the result is a differential form (not necessarily of top degree) on $\mathcal{M}_{g,n}$. This is an example of a correlation function in our conformal field theory. But this is not the most general example. Other correlation functions correspond to other local observables, such as the vector fields on M realized as Lie derivatives acting on differential forms.

Part of this structure is captured by the Gromov-Witten invariants. Since these moduli spaces $\mathcal{M}_{g,n}(M,\beta)$ are non-compact, we find that if we wish the correlation functions of Q–closed observables (such as closed differential forms on M) to depend only on their cohomology classes, we need to compactify these moduli spaces. The factorization property of the correlation functions will then also require that we introduce certain additional components into the compactified moduli spaces. The Kontsevich moduli spaces $\overline{\mathcal{M}}_{g,n}(M,\beta)$ of stable maps provide one with compactifications which satisfy all desirable properties and are equipped with the evaluation maps to the target

manifold M which one can use to pull-back differential forms on M. [4] One also has a forgetful map from $\overline{\mathcal{M}}_{g,n}(M,\beta)$ to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$. Taking the cup product of the pull-backs of such forms ω_i's to $\overline{\mathcal{M}}_{g,n}(M,\beta)$, and then the push-forward to $\overline{\mathcal{M}}_{g,n}$, we obtain differential forms on $\overline{\mathcal{M}}_{g,n}$ whose cohomology classes now depend only on the cohomology classes of the ω_i's. Pairing them with some natural cohomology classes on $\overline{\mathcal{M}}_{g,n}$, we obtain the Gromov-Witten invariants. But since they come from very special observables of our theory, they correspond to a particular sector of the full conformal field theory associated to the twisted sigma model in the infinite volume.

A natural question is how one can see the compactification $\overline{\mathcal{M}}_{g,n}(M,\beta)$ of $\mathcal{M}_{g,n}(M,\beta)$ in the framework of the conformal field theory with the action (1.3). A possible answer is that the integrals over the additional strata may naturally appear when one performs a regularization of the integral over the zero modes of the p_a's and $p_{\overline{a}}$'s along the lines described above.

Another part of this structure has been studied in mathematical literature starting with [26]. It is encoded by a sheaf of chiral algebras over M, called the chiral de Rham complex, which is defined by gluing the free chiral algebras on the overlaps of the open subsets. From the point of view of the sigma model, this chiral algebra corresponds to the cohomology of the right moving supercharge of the twisted sigma model in the *perturbative* regime (i.e., without counting instanton contributions), as explained in [34, 20]. However, the knowledge of this cohomology is not sufficient for determining the correlation functions of the sigma model. In order to determine them one needs to generalize the construction of this chiral algebra to the full conformal field theory and to include the instanton corrections. This is done in this paper in the case when the target manifold is a toric variety.

The idea is to realize the quantum field theory governed by the action (1.3) in the case when the target manifold M is a toric variety as a *deformation of a free field theory*. A toric variety $\mathbb{P}_S$ has a particularly nice open cover $\{\mathbb{A}_{\sigma(i)}\}_{i=1,\dots,N}$ with each open subset $\mathbb{A}_{\sigma(i)}$ isomorphic to $\mathbb{C}^d$ and their intersection $\mathbb{T}_S$ to $(\mathbb{C}^\times)^d$ (see Section 4.1). The complement of $\mathbb{T}_S$ in $\mathbb{P}_S$ is a divisor with components C_i equal to the complements of $\mathbb{A}_{\sigma(i)}$ in $\mathbb{P}_S$. Our idea is that the sigma model corresponding to a target manifold M is equivalent to a deformation of the sigma model with the target manifold $M \backslash C$, where C is a divisor, by means of a marginal vertex operator determined by C. Now, starting with the sigma model with the target $\mathbb{T}_S$, which is a free field theory, we may build the sigma models with the target manifolds obtained by gradually "gluing" back the divisors C_i. Each time we "glue" back a divisor C_i, we deform the theory by a vertex operator corresponding to C_i. Thus, the end result, which is the sigma model with the target $\mathbb{P}_S$, is identified with the deformation of the free field theory associated to $\mathbb{T}_S$ by means of the vertex operators corresponding to all $C_i, i = 1, \dots, N$. In this paper we identify these vertex operators and construct these deformations explicitly. Moreover, we use this description of the sigma model of $\mathbb{P}_S$ to give a new interpetation of mirror symmetry.

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[4]Note that it may happen that $\mathcal{M}_{g,n}(M,\beta)$ is empty, but $\overline{\mathcal{M}}_{g,n}(M,\beta)$ is non-empty; see the discussion at the end of Section 3.1.

We expect that one can give a similar description to the sigma models corresponding to more general target manifolds. A general complex manifold can be covered by open subsets that are analytically isomorphic to domains in $\mathbb{C}^n$. The supersymmetric sigma model corresponding to each of this open subsets is described by a free field theory which may be viewed as a system of decoupled bosonic and fermionic ghosts. So one may hope to define the quantum theory for a general Kähler target manifold M by appropriately "gluing" together the free field theories corresponding to these open subsets. The mathematical works on the chiral de Rham complex indicate that this is a non-trivial task which requires methods that up to now have not been widely used by physicists in this context, such as Cech cohomology. However, for toric varieties our task is considerably simplified by the existence of a particularly nice cover. We will use this cover in order to realize the sigma model as a deformation of a free field theory.

To illustrate these ideas, we will now consider the case when the target manifold M is $\mathbb{P}^1$.

1.4. **Warm-up example: From $\mathbb{C}$ to $\mathbb{P}^1$.** As a warm-up example, we will consider the case of the target manifold $M = \mathbb{P}^1$. The corresponding non-linear sigma model will be defined as a deformation of the linear model with the target $\mathbb{C}$. In the next section we will define the same non-linear model as a deformation of the linear model with the target $\mathbb{C}^\times$, which we will find to be technically more convenient. However, it is instructive to start by looking first at the deformation from $\mathbb{C}$ to $\mathbb{P}^1$.

The theory with the target $\mathbb{C}$ is a free conformal field theory with the chiral fields $X(z), p(z)$, $\psi(z)$, $\pi(z)$ and their anti-chiral partners with the action (0.1). The chiral fields obey the standard OPEs

$$p(z)X(w) = -\frac{i}{z-w} + \text{reg.}, \qquad \psi(z)\pi(w) = -\frac{i}{z-w} + \text{reg.}$$

This is nothing but the free theory of bosonic and fermionic ghosts (also known as a $\beta\gamma$–system and a bc–system), and its quantization is relatively straightforward.

We wish to interpret holomorphic maps $\Sigma \to \mathbb{P}^1$ within the framework of this free field theory. Namely, we view such maps as meromorphic maps $\Sigma \to \mathbb{C}$. Let $w_1, \ldots, w_n$ be the points of Σ where this map has a pole. Generically, all these poles will be of order one. As explained in the introduction, our proposal is that we can include such maps by inserting in the correlation functions of the linear sigma model certain vertex operators at the points $w_1, \ldots, w_n$. In the case at hand, we propose the following candidate for this operator:

$$D(z, \overline{z}) = \delta^2(p)(z, \overline{z})\pi(z)\overline{\pi}(\overline{z}).$$

What is the meaning of the operator $\delta^2(p)(z, \overline{z})$ from the Lagrangian point of view? Recall that the field $p(z)$ is a Lagrange multiplier responsible for the equation of holomorphy $\partial_{\overline{z}}X = 0$. Therefore the insertion of the field $\delta^2(p)(z, \overline{z})$ in the path integral is the instruction to relax this equation at the point $z \in \Sigma$ in the minimal possible way. This just means that our map $X : \Sigma \to \mathbb{P}^1$ should cease to be holomorphic at the point $z \in \Sigma$, i.e., it should develop a pole. We need to multiply $\delta^2(p)(z, \overline{z})$ by its odd counterpart, namely $\delta^2(\pi)(z, \overline{z})$, which is nothing but the operator $\pi(z)\overline{\pi}(\overline{z})$. This gives us the above operator $D(z, \overline{z})$. Inserting the operators $D(w_i, \overline{w}_i), i = 1, \ldots, n$, corresponds to considering meromorphic maps $\Sigma \to \mathbb{C}$ with poles precisely at the points $w_1, \ldots, w_n$, or

equivalently, considering the holomorphic maps $\Sigma \to \mathbb{P}^1$ which pass through the point $\infty \in \mathbb{P}^1$ precisely at the points $w_1, \ldots, w_n \in \Sigma$.

The operator $D(z, \overline{z})$ also has a transparent meaning from the point of view of the operator formalism. While operators of the form $\delta^2(X)(z, \overline{z})$ are quite common, the operators $\delta^2(p)(z, \overline{z})$ may appear at first glance as somewhat more exotic. But the mystery disappears if one considers the corresponding state in the Hilbert space of the linear sigma model corresponding to a small circle around a point $z \in \Sigma$. To simplify notation, set $z = 0$. Then this space contains the direct sum of the the tensor products

$$F_N \otimes \overline{F}_N, \qquad N \in \mathbb{Z},$$

of the Fock representations F_N the Heisenberg algebra generated by the Fourier modes of the chiral fields

$$X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n}, \qquad p(z) = \sum_{n \in \mathbb{Z}} p_n z^{-n-1},$$

and their anti-holomorphic analogues $\overline{F}_N$. The vacuum vector $|0\rangle \otimes \overline{|0\rangle}$ is in $F_0 \otimes \overline{F}_0$. The vector $|0\rangle \in F_0$ is annihilated by $\int X(z) f(z) dz$ for all holomorphic one-forms $f(z) dz$ on a small disc around 0, where the integral is taken over a small circle around 0 (i.e., it is annihilated by $X_n, n > 0$) and by $\int p(z) g(z) dz$, for all holomorphic functions $g(z)$ on the small disc around 0 (i.e., it is annihilated by $p_n, n \geq 0$). The vector $\overline{|0\rangle}$ satisfies similar equations.

Now, the vector corresponding to the operator $\delta^2(p)(0, 0)$ is nothing but the tensor product of the highest weight vectors from other Fock spaces, namely, $|1\rangle \otimes \overline{|1\rangle} \in F_1 \otimes \overline{F}_1$. The vector $|1\rangle$ satisfies

$$\int X(z) f(z) dz \cdot |1\rangle = 0, \qquad f(z) \in z\mathbb{C}[[z]],$$

$$\int p(z) g(z) dz \cdot |1\rangle = 0, \qquad g(z) \in z^{-1}\mathbb{C}[[z]].$$

In other words, $|1\rangle$ is annihilated by $X_n, n > 1$, and by $p_n, n \geq -1$. So

$$\delta^2(p)(z, \overline{z}) = \delta(p)(z) \delta(\overline{p})(\overline{z}),$$

where $\delta(p)(z)$ is nothing but the chiral field corresponding to the highest weight vector $|1\rangle$ of the Fock representation F_1 of the Heisenberg algebra, and $\delta(\overline{p})(\overline{z})$ is its anti-chiral analogue corresponding to the anti-chiral state $\overline{|1\rangle}$.

Likewise, $\pi_{-1}\overline{\pi}_{-1}|0\rangle$ is a highest weight vector over the Clifford algebra generated by the Fourier coefficients of the fields $\psi(z), \pi(z), \overline{\psi}(z), \overline{\pi}(z)$. It is annihilated by $\psi_n, \overline{\psi}_n, n > 1$, and $\pi_n, \overline{\pi}_n, n \geq -1$.

Incidentally, from this point of view $\delta^2(X)(z, \overline{z})$ is nothing but the operator corresponding to the state $|-1\rangle \otimes \overline{|-1\rangle}$. So the familiar operator

$$\mathcal{O}_0(z, \overline{z}) = \delta^2(X)(z, \overline{z}) \psi(z) \overline{\psi}(\overline{z})$$

is an analogue of our operator $D(z, \overline{z})$, which may in fact be used to represent the observable in the Gromov-Witten theory corresponding to the degree two cohomology class of $\mathbb{P}^1$.

The conformal dimension of the field $\delta^2(p)$ is $(-1,-1)$. This is in fact a special case of a general fact: if $\Phi(z,\overline{z})$ is a bosonic field of conformal dimension $(\Delta, \overline{\Delta})$ and charge ν, then $\delta^2(\Phi)(z,\overline{z})$ should have conformal dimension $(-\Delta, -\overline{\Delta})$ and charge $-\nu$. Note also that $D(z,\overline{z})$ has conformal dimension $(0,0)$.

Let us compute the correlation function of these observables for Σ of genus zero. From the Gromov-Witten theory we know that the correlation function is non-zero if the number of insertions is odd, $2n + 1$, and then the answer should be equal to q^n, because it corresponds to holomorphic maps of degree n. Let us explain how to reproduce exactly this answer within the framework of the linear sigma model. Observe that a map of degree n has to pass through ∞ exactly n times (with multiplicities, in general, but generically the multiplicities will all be equal to one). This means that we have to insert the operator $D(z,\overline{z})$ at n distinct points. But this operator has charge 1 (with respect to the current $:X(z)p(z):$) and ghost number 1, while the operator $\mathcal{O}(z,\overline{z})$ has charge -1 and ghost number -1. The anomalous conservation law in genus zero demands that the total charge and the ghost number be both equal to -1. Therefore in order to compensate for the n insertions of the operator $D(z,\overline{z})$ we have to insert the operator $\mathcal{O}(z,\overline{z})$ at $n+1$ additional points. After that we reproduce the answer of the Gromov-Witten theory because the correlation function of these operators is equal to 1, which we should multiply by q^n to account for the degree of the map. In other words, in order to account for the degrees of the holomorphic maps we should really be inserting the operator $qD(z,\overline{z})$ rather than $D(z,\overline{z})$.

In the Gromov-Witten theory one also considers the fields obtained by cohomological descent from the basic fields described above (see [31]). The cohomological descendants of an operator $\mathcal{O}$ satisfy the equations

$$d\mathcal{O} = [Q_{\text{tot}}, \mathcal{O}^{(1)}], \qquad d\mathcal{O}^{(1)} = [Q_{\text{tot}}, \mathcal{O}^{(2)}],$$

where $Q_{\text{tot}} = Q + \overline{Q}$ is the supersymmetry charge. To calculate them, we observe that we have two (twisted) $N = 2$ superconformal algebras with the chiral one generated by the fields

$$G(z) = i\partial_z X(z)\pi(z), \qquad Q(z) = -ip(z)\psi(z),$$

$$T(z) = -i:\partial_z X(z)p(z): - i:\pi(z)\partial_z\psi(z):, \qquad J(z) = i:\psi(z)\pi(z):,$$

and similarly for the anti-chiral one. The chiral supersymmetry charge is the operator $Q = \int Q(z)dz$, and $G(z)$ satisfies

$$\int Q(w)dw \cdot G(z) = T(z)$$

(here and below, in similar formulas, the contour of integration goes around z, and we suppress the factor $1/2\pi i$). In particular, we have

$$[Q, G_{-1}]_+ = L_{-1},$$

where $G_{-1} = \int G(z)dz$. We have similar formulas for $\overline{Q}$.

This allows us to find $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$ from the formulas

$$\mathcal{O}^{(1)} = G_{-1} \cdot \mathcal{O}dz + \overline{G}_{-1} \cdot \mathcal{O}d\overline{z}, \qquad \mathcal{O}^{(2)} = G_{-1}\overline{G}_{-1} \cdot \mathcal{O}dzd\overline{z},$$

provided that $Q_{\mathrm{tot}} \cdot \mathcal{O} = 0$. In particular, since $Q_{\mathrm{tot}} \cdot D(z, \overline{z}) = 0$, we find that

$$D^{(2)}(z, \overline{z}) = \left(\int X(w) w dw \int \overline{X}(\overline{w}) \overline{w} d\overline{w} \cdot \delta^2(p)(z, \overline{z}) \right) \pi(z) \partial_z \pi(z) \overline{\pi}(\overline{z}) \partial_{\overline{z}} \overline{\pi}(\overline{z}) dz d\overline{z}.$$

The bosonic part of this field corresponds to the state $X_1 |1\rangle \otimes \overline{X}_1 \overline{|1\rangle} \in F_1 \otimes \overline{F}_1$. Note that the field $D^{(2)}(z, \overline{z})$ has conformal dimension $(1, 1)$.

In the setting of the linear sigma model the maps $\Sigma \to \mathbb{P}^1$ of degree n are the same as meromorphic maps with poles at n points (counted with multiplicity). As we argued above, those should be counted via the insertion of the vertex operator $qD(z, \overline{z})$. Since we will be integrating over all such maps, and hence over all possible positions of the poles, the degree n contribution to the correlation function $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\mathbb{P}^1}$ of local observables in the non-linear sigma model with the target $\mathbb{P}^1$ (such as $\mathcal{O}_0(z, \overline{z})$ introduced above) should be equal to the correlation function of these operators in the linear model with the additional insertion of the integral of the $(1, 1)$–forms $qD^{(2)}(z, \overline{z})$, obtained by cohomological descent from the operators $D(z, \overline{z})$ introduced above. Thus, this correlation function should be given by

$$\sum_{n=0}^{\infty} \frac{q^n}{n!} \langle \mathcal{O}_1 \ldots \mathcal{O}_m \int D^{(2)}(w_1, \overline{w}_1) \ldots \int D^{(2)}(w_1, \overline{w}_n) \rangle_{\mathbb{C}}$$

(the $1/n!$ factor is due to the fact that the points $w_1, \ldots, w_n$ are unordered). But this is the same as the correlation function in the linear sigma model deformed by the marginal operator $qD^{(2)}(z, \overline{z})$.

This suggests that *the non-linear sigma model with the target $\mathbb{P}^1$ in the infinite volume limit is equivalent to the linear sigma model deformed by the marginal operator* $D^{(2)}(z, \overline{z})$, i.e., the theory defined by the action

$$\frac{1}{2\pi} \int_{\Sigma} d^2 z \, \left(i p \partial_{\overline{z}} X + i \pi \partial_{\overline{z}} \psi + i \overline{p} \partial_z \overline{X} + i \overline{\pi} \partial_z \overline{\psi} + q D^{(2)} \right).$$

While nice and intuitive, the representation of the deforming operator in terms of the delta-function $\delta^2(p)(z, \overline{z})$ is rather inconvenient for practical calculations. One possible way to do that is to invoke the Friedan-Martinec-Shenker bosonization [15] of the p, X system:

$$X(z) = e^{u(z) + v(z)}, \qquad p = -\partial v(z) e^{-u(z) - v(z)},$$

where $u(z)$ and $v(z)$ are the scalar fields having the OPEs

$$u(z) u(w) \sim -\log(z - w), \qquad v(z) v(w) \sim \log(z - w).$$

We have similar formulas for the anti-chiral fields $\overline{p}, \overline{X}$. Then we have the following bosonic representation

$$\delta(p)(z) = e^{u(z)}, \qquad \delta(\overline{p})(\overline{z}) = e^{\overline{u}(\overline{z})}.$$

It is easy to see that these fields have the right OPE with the fields $X(z), p(z)$ and their complex conjugates. Since the conformal dimension of $e^{\alpha u(z)}$ is $-\alpha(\alpha + 1)/2$, we obtain that the conformal dimension of $\delta(p(z))$ is indeed -1.

Thus, we obtain the following realization of the fields introduced above:

$$D(z, \overline{z}) = e^{u(z) + \overline{u}(\overline{z})} \pi(z) \overline{\pi}(\overline{z}),$$

$$D^{(2)}(z, \overline{z}) = e^{2(u + \overline{u}) + (v + \overline{v})} \pi \partial_z \pi \overline{\pi} \partial_{\overline{z}} \overline{\pi} dz d\overline{z}.$$

However, the FMS bosonization identifies the X, p system with a subalgebra of the chiral algebra of the two scalar bosons u, v. To get an isomorphism, we need to invert X, i.e., pass from $\mathbb{C}$ to $\mathbb{C}^{\times}$ (see [11]). This already indicates that it is more convenient to formulate the theory on $\mathbb{C}^{\times}$ rather than on $\mathbb{C}$. This leads us to the toric sigma model introduced in the next section.

2. The model with the target $\mathbb{C}^{\times}$

2.1. **Toric sigma model.** We would like to express the correlation functions of the sigma model with the target $\mathbb{P}^1$ in the limit of infinite volume in terms of the operator formalism of the sigma model with the target $\mathbb{C}^{\times} = \mathbb{P}^1 \backslash \{0, \infty\}$, also at the infinite volume.

To define the sigma model with the target $\mathbb{C}^{\times}$ we will use the logarithmic coordinate $X = R + i\phi$, where ϕ is periodic with the period 2π. In other words, we identify $\mathbb{C}^{\times}$ with $\mathbb{R} \times S^1$, where R is a coordinate on $\mathbb{R}$ and ϕ is a coordinate on S^1. We introduce the metric

$$(2.1) \qquad t(dR^2 + d\phi^2) = t dX d\overline{X},$$

so the circle has radius $\sqrt{t}$. The action of the sigma model in the first order formalism, introduced in Section 1.1, is

$$(2.2) \qquad I_t = \frac{1}{2\pi} \int_{\Sigma} d^2 z \; \left(ip \partial_{\overline{z}} X + i\overline{p} \partial_z \overline{X} + i\pi \partial_{\overline{z}} \psi + i\overline{\pi} \partial_z \overline{\psi} + t^{-1} p \overline{p} \right).$$

To eliminate p and $\overline{p}$ in the path integral by completing the action to a square and integrating them out (see Remark 1.1), we substitute the following expressions in the Lagrangian:

$$(2.3) \qquad p = -it \partial_z \overline{X}, \qquad \overline{p} = -it \partial_{\overline{z}} X.$$

Then we obtain the usual action of the sigma model with the target $\mathbb{R} \times S^1$ with the metric (2.1):

$$\frac{t}{2\pi} \int_{\Sigma} d^2 z \; \left(\partial_{\overline{z}} X \partial_z \overline{X} + i\pi \partial_{\overline{z}} \psi + i\overline{\pi} \partial_z \overline{\psi} \right).$$

In the limit $t \to \infty$ the last term in I_t drops out and we obtain the action

$$(2.4) \qquad \frac{i}{2\pi} \int_{\Sigma} d^2 z \; \left(p \partial_{\overline{z}} X + \pi \partial_{\overline{z}} \psi + \overline{p} \partial_z \overline{X} + \overline{\pi} \partial_z \overline{\psi} \right).$$

We call this model a *toric sigma model* with the target $\mathbb{C}^{\times}$.

Equations of motion imply that fields $X(z), p(z), \psi(z), \pi(z)$ are holomorphic ($X(z)$ and $\psi(z)$ have conformal dimension 0 and $p(z), \pi(z)$ have conformal dimension 1), while their complex conjugates $\overline{X}(\overline{z}), \overline{p}(\overline{z}), \overline{\psi}(\overline{z}), \overline{\pi}(\overline{z})$ are anti-holomorphic. They obey the standard OPEs

$$(2.5) \qquad X(z) p(w) = -\frac{i}{z - w} + :X(z) p(w):, \qquad \psi(z) \pi(w) = -\frac{i}{z - w} + :\psi(z) \pi(w):,$$

and similarly for the anti-chiral fields. So this is a free field theory which is the toric version of the well-known system of bosonic and fermionic ghost fields. It possesses an $N = 2$ superconformal symmetry. The generating fields of the left moving $N = 2$ (twisted) superconformal algebra are given by the following formulas:

$$(2.6) \qquad Q(z) = -ip(z)\psi(z) - \partial_z\psi(z), \qquad G(z) = i\partial_z X(z)\pi(z),$$

$$T(z) = -i{:}\partial_z X(z)p(z){:} - i{:}\pi(z)\partial_z\psi(z){:}, \qquad J(z) = i{:}\psi(z)\pi(z){:} + \partial_z X(z).$$

There are also anti-chiral fields $\overline{Q}(\overline{z}), \overline{G}(\overline{z}), \overline{T}(\overline{z})$, and $\overline{J}(\overline{z})$, given by similar formulas, which generate the right moving copy of the $N = 2$ superconformal algebra.

The Hilbert space of the theory is built from bosonic Fock representations of the Heisenberg algebra generated by the Fourier coefficients of the fields $\partial_z X(z), p(z)$, $\partial_{\overline{z}} X(\overline{z}), \overline{p}(\overline{z})$ and fermionic Fock representations of the Clifford algebra generated by the Fourier coefficients of $\psi(z), \pi(z), \overline{\psi}(\overline{z}), \overline{\pi}(\overline{z})$. The precise structure of the bosonic Hilbert space and the state-field correspondence will be described in Section 5.1. Here we focus on the most salient features of the theory.

2.2. Holomorphic vortices.
In the canonical quantization of the toric sigma model we consider the theory defined on the cylinder Σ, with the holomorphic coordinate $z = e^{t+is}, t \in \mathbb{R}, s \in \mathbb{R}/2\pi\mathbb{Z}$. Because our target space is also a cylinder, we find that we can allow non-trivial winding, i.e., we can allow $X(e^{2\pi i}z)$ to differ from $X(z)$ by a an integral multiple of $2\pi i$. This, together with the condition of holomorphy, means that $X(z)$ and $\overline{X}(\overline{z})$ may be written as follows:

$$X(z) = \omega \log z + \sum_{n\in\mathbb{Z}} X_n z^{-n}, \qquad \overline{X}(\overline{z}) = \omega \log \overline{z} + \sum_{n\in\mathbb{Z}} \overline{X}_n \overline{z}^{-n},$$

where ω is the *winding operator* which is allowed to take integer values. This indicates that the Hilbert space may contain states that have non-zero value of the operator ω, and hence non-zero winding.

A convenient way to understand this is by interpreting the toric sigma model as the $\mathbb{Z}$–orbifold of the corresponding model with the target $\mathbb{C}$. The latter is the free field theory that we discussed in Section 1.4. It is described by the action (2.4), where how $X(z)$ and $\overline{X}(\overline{z})$ are single-valued. The group $\mathbb{Z}$ is a symmetry group of the action, shifting X by integer multiples of $2\pi i$. We expect that our toric sigma model with the target $\mathbb{C}/2\pi i\mathbb{Z}$ may be obtained from the corresponding theory with the target $\mathbb{C}$ by taking its $\mathbb{Z}$–orbifold. The corresponding twist fields should then be exactly the fields with non-zero winding number ω.

This is analogous to the fact that the vortex operators of the sigma model with the target $\mathbb{C}/2\pi i\mathbb{Z}$ at the finite radius may be interpreted as the twist fields arising in the $\mathbb{Z}$–orbifolding of the usual linear sigma model with the target $\mathbb{C}$. Because of this analogy, we call the twist fields arising in the toric sigma model *holomortex operators*. However, the vortex operators and the twist fields that we have at the infinite radius have different nature.

To explain this point, it is convenient to work in the logarithmic coordinates s, t on the worldsheet cylinder Σ_0. In the finite volume theory the coordinates R, ϕ on the target cylinder M_0 are completely independent, and the winding occurs in the ϕ

variable, independently of R. In other words, there are harmonic maps $X : \Sigma_0 \to M_0$ which are constant along R, but wind around ϕ, such as $R = 0, \phi = ms$, where $m \in \mathbb{Z}$. The vortex operator with the winding number m belongs to the sector of the theory corresponding to maps of this type. But in the infinite volume limit the map $X : \Sigma \to M$ has to be *holomorphic.* Therefore R and ϕ are no longer independent. Now we have maps of the form $R = mt, \phi = ms$, where $m \in \mathbb{Z}$, so R as well as ϕ depend on (s, t). That is why there is no straightforward way to define the holomorphic winding operators as a naive limit of the vortex operators in the infinite volume limit.

What are then the explicit formulas for the holomortex operators? Denoting the operator with the winding number m by $\Psi_m(z, \overline{z})$, we find that we need to have the following OPEs:

$$(2.7) \qquad X(z)\Psi_m(w, \overline{w}) = m \log(z - w)\Psi_m(w, \overline{w}) + ...,$$
$$\overline{X}(\overline{z})\Psi_m(w, \overline{w}) = m \log(\overline{z} - \overline{w})\Psi_m(w, \overline{w}) + ...$$

Using the OPEs (2.5), we find that the field

$$(2.8) \qquad \Psi_m(z, \overline{z}) = e^{-im \int P}(z, \overline{z}) = \exp\left(-im \int_{z_0}^{z} (p(w)dw + \overline{p}(\overline{w})d\overline{w})\right)$$

(or any of its scalar multiples) has precisely the OPEs (2.7) with $X(z)$ and $\overline{X}(\overline{z})$.

Formula (2.8) a priori depends on the point z_0 and the integration contour. We will give a more precise definition of these operators acting on the Hilbert space of the theory in Section 5.1. Here we would like to comment that for the purposes of this paper we only need to consider the correlation functions of the operators $e^{\pm i \int P}$. We will postulate that a correlation function of such operators will be non-zero if and only if an equal number of these operators with the $+$ and $-$ signs are involved. (In Section 2.3 we will see that this condition naturally comes from integrating over the zero mode of the dual variable U.) Then we simply define the correlation function by pairing the $+$ and $-$ operators in an arbitrary way and integrating over the contours going from the location of the $-$ operator to the location of the $+$ operator in each pair. The result is independent of the choice of the pairing as long as all other operators in the correlation function have well-defined OPEs with the operators $e^{\pm i \int P}$, as discussed below. Note also that while the individual operator $e^{\pm i \int P}$ is a priori defined only up to a scalar multiple, once we normalize one of them, the other is also automatically normalized. Therefore the product of an equal number of the $+$ and $-$ holomortex operators does not depend on the choice of normalization. This gives us a well-defined prescription for the computation of the correlation functions that we need, and it is easy to generalize it to the correlation functions involving the fields Ψ_m with $m \neq \pm 1$.

The presence of the holomortex operators $e^{im \int P}, m \in \mathbb{Z}$, given by formula (2.8), in our theory places restrictions on what other fields are allowed. Namely, those fields must have well-defined OPEs with the fields $e^{im \int P}, m \in \mathbb{Z}$. (This insures the contour independence of the correlation functions discussed in the previous paragraph.) This is analogous to the case of the sigma model with the target $\mathbb{C}/2\pi i\mathbb{Z}$ at the finite radius, considered as a $\mathbb{Z}$–orbifold. In the linear sigma model we have the fields $e^{ir\phi}$ with arbitrary $r \in \mathbb{R}$, but after orbifolding r is quantized and can take only integer values. This

condition insures that these fields have well-defined OPEs with the vortex operators, which are the orbifold twist fields at the finite radius.

Let us analyze what conditions are imposed by the presence of the twist fields $e^{im \int P}, m \in \mathbb{Z}$ in our theory. The fields $p(z), \overline{p}(\overline{z})$ have well-defined OPEs with them, and so do the derivatives $\partial_z X(z), \partial_{\overline{z}} \overline{X}(\overline{z})$. Next, we look at the exponential fields $\exp(\alpha X(z) + \beta \overline{X}(\overline{z}))$. They have the following OPEs with $e^{-im \int P}(w, \overline{w})$:

$$\exp(\alpha X(z) + \beta \overline{X}(\overline{z})) e^{-im \int P}(w, \overline{w}) =$$
$$(z - w)^{m\alpha} (\overline{z} - \overline{w})^{m\beta} {:} \exp(\alpha X(z) + \beta \overline{X}(\overline{z})) e^{-im \int P}(w, \overline{w}) {:} \, .$$

The condition for the right hand side to be single-valued is that $\alpha - \beta \in \mathbb{Z}$. This condition ensures that the correlation functions of the allowed operators and the operators $e^{-im \int P}(w, \overline{w})$ do not depend on the choice of the contours of integration. The operator content of the theory is described in more detail in Section 5.1.

2.3. T–**duality.** Now we will show that the toric sigma model introduced in the previous section is equivalent to the ordinary sigma model with the target space being the torus $\mathbb{R} \times S^1$ equipped with the *Minkowski* metric such that the circle is *isotropic*. In this realization the holomortex operators $e^{im \int P}$ have a particularly simple form.

In this section we discuss the path integral realization of the duality. The operator realization will be considered in Section 5.2. Let us introduce the one-form

$$P = p(z) dz + \overline{p}(\overline{z}) d\overline{z}$$

on Σ. We choose the real structure in which the complex conjugate of p is $\overline{p}$, so that the one-form P is real. Then we rewrite the bosonic part of the action (2.4) as follows:

$$(2.9) \qquad\qquad I_{\text{bos}} = \frac{i}{2\pi} \int_\Sigma (-P \wedge d\phi + P \wedge *dR).$$

Here $*$ denotes the Hodge star operator on Σ, which in coordinates looks as follows: $*dz = -i dz, *d\overline{z} = i d\overline{z}$. Recall that our convention for the integration measure on Σ is $d^2 z = i dz \wedge d\overline{z}$.

Let us integrate out the field ϕ in the path integral. Then we obtain the constraint $dP = 0$, or in components $\partial_z \overline{p} = \partial_{\overline{z}} p$. A general solution of this equation is

$$(2.10) \qquad\qquad P = dU = dU_0 + \sum_{j \in I} a_j \omega_j,$$

where U_0 is a real single-valued field and the ω_i's are closed real one-forms representing a basis in the first cohomology group of Σ. We choose them in such a way that they are harmonic and their integrals over cycles in Σ are integers and $J^{kl} = \int \omega_k \wedge \omega_l$ is an integral skew-symmetric matrix with determinant one. We claim that the coefficients a_i are constrained to be of the form $a_j = 2\pi m_j, m_j \in \mathbb{Z}$, and so U is a 2π–periodic field.

We follow the presentation of the book [19], Sect. 11.2. The field ϕ takes values in $\mathbb{R}/2\pi\mathbb{Z}$ and therefore it is allowed to have non-trivial winding. This means that $d\phi$ may

be expressed by the formula

$$d\phi = d\phi_0 + 2\pi \sum_{i \in I} n_i \omega_i, \qquad n_i \in \mathbb{Z},$$

where ϕ_0 is a real single-valued function. Then we have

$$\frac{1}{2\pi} \int_\Sigma P \wedge d\phi = \sum_{i,j \in I} a_i J^{ij} n_j.$$

Taking the summation over the n_j's in the path integral, we find from the Poisson summation formula that $a_j = 2\pi m_j, m_j \in \mathbb{Z}$. Hence U is a function $\Sigma \to \mathbb{R}/2\pi\mathbb{Z}$.

Thus, we have the following transformation formulas:

$$p(z) = \partial_z U(z, \overline{z}), \qquad \overline{p}(\overline{z}) = \partial_{\overline{z}} U(z, \overline{z}),$$

$$\frac{1}{2}(X(z) + \overline{X}(\overline{z})) = R(z, \overline{z}).$$

These formulas are closely related to the Friedan-Martinec-Shenker bosonization discussed in Section 1.4. The holomortex operators $e^{im \int P}$ have a particularly simple realization in the dual variables:

$$e^{im \int_{z_0}^z P} = e^{imU(z)} e^{-imU(z_0)},$$

and this is the reason why the dual theory will be convenient for our purposes.

Let us introduce the improved holomortex operators $e^{imU(z)}$. The integration over the zero mode of the field $U(z)$ will guarantee that the correlation function of the operators $e^{\pm iU(z)}$ will be non-zero if and only if equal numbers of the operators $e^{iU(z)}$ and $e^{-iU(z)}$ are involved. This is precisely the condition that we imposed by hand in Section 2.2.[5] On the other hand, if this condition is satisfied, then the correlation functions of the improved holomortex operators are the same as the correlation functions of the original ones. Hence from now on we will use the improved holomortex operators in our computations.

The dual theory is formulated in terms of the fields U and R with the action

$$(2.11) \qquad \widetilde{I}_{\text{bos}} = \frac{i}{2\pi} \int_\Sigma dU \wedge *dR = \frac{i}{2\pi} \int_\Sigma d^2z \, (\partial_z U \partial_{\overline{z}} R + \partial_{\overline{z}} U \partial_z R).$$

This is the action of the sigma model with the target the cylinder $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ with coordinates (R, U), but with the *Minkowski* metric $idRdU$. Note that the compact direction U is isotropic, and so the notion of the "radius" of this cylinder does not make sense.

Since the one-forms ω_j's in formula (2.10) are chosen to be harmonic, we can replace in the action the multivalued function U by the single-valued function U_0. However, we then have to remember to integrate in the path integral not only over U_0 but also sum up over all possible values of $a_j = 2\pi m_j, m_j \in \mathbb{Z}$. Because our metric is Minkowski and the dual variable to U, namely R, is non-periodic, this leads to some non-trivial consequences as discussed below in Section 3.2.

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[5]We thank V. Lysov for a discussion of this point.

The fermionic part of action of the theory remains the same, so the total action of the dual theory is

$$(2.12) \qquad \widetilde{I} = \frac{i}{2\pi} \int_\Sigma d^2z \, (\partial_z U \partial_{\overline{z}} R + \partial_{\overline{z}} U \partial_z R + \pi \partial_{\overline{z}} \psi + \overline{\pi} \partial_z \overline{\psi}).$$

Note that when we deform the action (2.4) to a finite radius r, we add to it the term $\frac{1}{r^2} p \overline{p}$, which in the dual variables looks as $\frac{1}{r^2} \partial_z U \partial_{\overline{z}} U$. Therefore we see that the metric on the torus is changing in such a way that the circle in the U direction acquires radius r^{-1}, as we should expect under T–duality at the finite radius.

3. Changing the target from $\mathbb{C}^\times$ to $\mathbb{P}^1$

The correlation functions of the toric sigma model correspond to path integrals over all maps $\Sigma \to \mathbb{C}^\times$. Then, since the path integral over p and π and their complex conjugates is interpreted as the delta-form supported on the holomorphic maps, as we argued above, any correlation function of the fields involving $X(z)$ and $\psi(z)$ (and their complex conjugates) may be written in terms of the holomorphic maps $\Sigma \to \mathbb{C}^\times$, which are necessarily constant for compact Σ. Therefore the correlation functions reduce to integrals over the zero mode (i.e., over the image of the constant map $\Phi : \Sigma \to \mathbb{C}^\times$). Is it possible to interpret holomorphic maps $\Sigma \to \mathbb{P}^1$ within the framework of the toric sigma model?

3.1. **Deformation of the toric sigma model.** As we explained in the Introduction, holomorphic maps $\Sigma \to \mathbb{P}^1$ may be viewed as holomorphic maps $\Sigma \backslash \{w_i^\pm\} \to \mathbb{C}/2\pi i \mathbb{Z}$ with logarithmic singularities at some points $w_1^\pm, \ldots, w_N^\pm$ where this map behaves as $\pm \log(z - w_i^\pm)$. These singular points correspond to zeroes and poles of $\exp \Phi$, and generically they will be distinct. Our proposal is that *we can include these maps by inserting in the correlation function of the linear sigma model certain vertex operators* $\Psi_\pm(w_i^\pm)$.

The defining property of the operators $\Psi_\pm(w)$ is that their operator product expansion (OPE) with $X(z)$ should read

$$X(z)\Psi_\pm(w) = \pm \log(z - w)\Psi_\pm(w).$$

We have already found such operators in Section 2.2. These are the holomortex operators

$$\Psi_\pm(w) = e^{\mp i \int_{w_0}^w P}.$$

Note that using these operators we can obtain a given function (for Σ of genus zero)

$$\Phi(z) = c + \sum_{i=1}^n \log(z - w_i^+) - \sum_{i=1}^n \log(z - w_i^-)$$

as the correlator

$$(3.1) \qquad \Phi(z) = \langle X(z) \prod_{i=1}^n \Psi_+(w_i^+) \prod_{i=1}^n \Psi_-(w_i^-) \left(\delta^2(X(z_0) - c_0)\psi(z_0)\overline{\psi}(\overline{z}_0) \right) \rangle.$$

The factor in brackets is needed so as to normalize the function $\Phi(z)$ by the condition that $\Phi(z_0) = c_0$. This condition naturally appears upon integrating over the zero modes of X and ψ.

Therefore we can *create* any meromorphic function on Σ of genus 0 by taking the correlation function of the form (3.1). How to generalize this to Σ of genus greater than zero? In this case, for a meromorphic function to exist, the points $w_i^{\pm}$ where it has zeroes and poles must satisfy a constraint: the divisor $\sum_i (w_i^+) - \sum_i (w_i^-)$ has to be in the kernel of the Abel-Jacobi map. Therefore for our theory to be consistent, the correlation functions must somehow take this condition into account.

This appears puzzling at first, but the apparent paradox is resolved if we recall formula (2.10). The one-form P is defined up to an addition of a linear combination of closed one-forms ω_j, and periodicity of the field ϕ implies that the coefficients a_j in front of these one-forms must be integer multiples m_j's of 2π. In the path integral we need to sum up over the m_j's, and this leads to non-trivial consequences.

Let $\mathcal{O}_i, i = 1, \ldots, N$, be some local operators in the toric sigma model and suppose we wish to compute the correlation function of these as well as the holomortex operators $e^{-i \int^{w_j^+} P}, j = 1, \ldots, n_+$, and $e^{i \int^{w_j^-} P}, j = 1, \ldots, n_-$. First of all, recall from Section 2.2 that the number of insertions of $e^{-i \int P}$ has to be equal to the number of insertions of $e^{i \int P}$; otherwise, the correlation function is automatically zero. Thus, $n_+ = n_- = n$. The correlation function should be a differential form on the moduli space $\mathcal{M}_{g,N+2n}$. To simplify our analysis, let us fix the complex structure on Σ and the positions of the operators $\mathcal{O}_i, i = 1, \ldots, N$, leaving the positions $w_j^{\pm}$ of the holomortex operators free, but distinct. Consider the resulting differential form ω on the configuration space of $2n$ distinct points on Σ. Its degree is determined by the fermionic charge conservation. Assume for simplicity that the operators $\mathcal{O}_i$ do not contain fermions. As in genus zero, we need to insert an operator of the form $\delta^2(X(z_0) - c_0)\psi(z_0)\overline{\psi}(\overline{z}_0)$ to take care of the zero mode of X and ψ. This means that in addition we need to insert g operators π and $\overline{\pi}$, so that we should get a (g, g)–form on the configuration space.

According to a general prescription of [33, 36] (see also Section 1.3 above), this differential form is constructed as follows. It is completely determined by its values on g tangent vectors of the form $\partial/\partial w_j^{\pm}$ and g tangent vectors of the form $\partial/\partial \overline{w}_j^{\pm}$. Then at the corresponding point we have to insert the operators $G_{-1}\Psi_{\pm}$, $\overline{G}_{-1}\Psi_{\pm}$ or $G_{-1}\overline{G}_{-1}\Psi_{\pm}$. For example, let j run from 1 to g. Then at the points w_j^+ we have to insert the operator $e^{-i \int^{w_j^+} P} \pi(w_j^+)\overline{\pi}(\overline{w}_j^+)$. The corresponding value of our differential form ω is given by

$$\langle \prod_{i=1}^N \mathcal{O}_i \prod_{j=1}^g e^{-i \int^{w_j^+} P} \pi(w_j^+)\overline{\pi}(\overline{w}_j^+) \prod_{j=g+1}^n e^{-i \int^{w_j^+} P} \prod_{j=1}^n e^{i \int^{w_j^-} P} \rangle$$

$$= \langle \prod_{i=1}^N \mathcal{O}_i \prod_{j=1}^g \pi(w_j^+)\overline{\pi}(\overline{w}_j^+) \prod_{j=1}^n e^{i \int_{w_j^+}^{w_j^-} P} \rangle.$$

Substituting formula (2.10), we find that the correlation function will contain the factor

$$\exp\left(2\pi i \sum_k \sum_{j=1}^n m_k \int_{w_j^+}^{w_j^-} \omega_k\right),$$

and this is the only term that depends on the m_k's. In the path integral we will have to take the sum over all values of the m_k's. The result of this summation is a delta-function, which means that the correlation function is identically equal to zero unless

$$\sum_{j=1}^n m_k \int_{w_j^+}^{w_j^-} \omega_k = 0$$

for all k. This precisely means that the divisor $\sum_j (w_j^+) - \sum_j (w_j^-)$ has to be in the kernel of the Abel-Jacobi map.

Now it is clear that the differential form ω on the configuration space of $2n$ points that we obtain in our theory is the delta-form supported on the kernel of the Abel-Jacobi map (which has codimension g). We can "smoothen" this delta-form by deforming the action of our model with the term $\epsilon \int_\Sigma p\overline{p}d^2z$ (see below).

Now suppose that $\mathcal{O}_i, i = 1, \ldots, N$, are operators from the sigma model with the target $\mathbb{P}^1$, and we wish to compute the correlation function in the sigma model with the target $\mathbb{P}^1$

(3.2) $$\langle \mathcal{O}_1 \ldots \mathcal{O}_N \rangle_{\mathbb{P}^1} = \sum_{n \geq 0} \langle \mathcal{O}_1 \ldots \mathcal{O}_N \rangle_{\mathbb{P}^1, n} q^n,$$

where $\langle \mathcal{O}_1 \ldots \mathcal{O}_N \rangle_{\mathbb{P}^1, n}$ is the term corresponding to the holomorphic maps of degree n. As we explained above, more general correlation functions in our sigma model are obtained by inserting contour integrals of the fields $G(z)$ and $\overline{G}(\overline{z})$ coupled to vector fields on $\mathcal{M}_{g,n}$. These correlation functions are interpreted as differential forms on $\mathcal{M}_{g,n}$.

As we discussed above, in the setting of the linear sigma model with the target $\mathbb{C}^\times$ the maps of degree n are maps with logarithmic singularities at $2n$ points $w_j^\pm, j = 1, \ldots, n$ (counted with multiplicity). For fixed positions of these points such a map is counted by inserting in the correlation function the holomortex operators $\Psi_\pm(w_j^\pm, \overline{w}_j^\pm)$. Including *all* possible positions of the points $w_j^\pm$ means applying to each field $\Psi_\pm(w_j^\pm, \overline{w}_j^\pm)$ the operator $G_{-1}\overline{G}_{-1}$, where G_{-1} and $\overline{G}_{-1}$ are the contour integrals of the fields $G(z)$ and $\overline{G}(\overline{z})$, coupled to the translation vector field $\partial/\partial w_j^\pm$ and $\partial/\partial \overline{w}_j^\pm$, respectively. In other words, we must replace each field $\Psi_\pm(w, \overline{w})$ by the corresponding $(1,1)$–form $\Psi_\pm^{(2)}(w, \overline{w})$ given by the formula

$$\Psi_\pm^{(2)}(w, \overline{w}) = G_{-1}\overline{G}_{-1} \cdot \Psi_\pm(w, \overline{w})dwd\overline{w} = e^{\mp i \int^w P} \pi(w)\overline{\pi}(\overline{w})dwd\overline{w},$$

and integrate these $(1,1)$–forms over Σ. Note that since the operators $\Psi_\pm$ are Q–closed, $\Psi_\pm^{(2)}$ is the operator obtained by cohomological descent (see Section 1.4).

Thus, we find that

$$\langle \mathcal{O}_1(z_1, \overline{z}_1) \dots \mathcal{O}_N(z_N, \overline{z}_N) \rangle_{\mathbb{P}^1, n} = \frac{1}{(n!)^2} \langle \mathcal{O}_1(z_1, \overline{z}_1) \dots \mathcal{O}_N(z_N, \overline{z}_N)$$

$$\times \prod_{i=1}^{n} \int_{\Sigma} \Psi_+^{(2)}(w_j^+, \overline{w}_j^+) \prod_{i=1}^{n} \int_{\Sigma} \Psi_-^{(2)}(w_j^-, \overline{w}_j^-) \rangle_{\mathbb{C}^\times}$$

(the coefficient $1/(n!)^2$ is due to the fact that the collections of points $\{w_j^+\}$ and $\{w_j^+\}$ are unordered).

The integrand is not well-defined on the diagonals $w_i^+ = w_j^-$ near the points z_k, a typical singularity being $|z_k - w_i^+|^2/|z_k - w_-^j|^2$. However, we believe that the above integrals do converge as long as we choose smooth observables $\mathcal{O}_i$ (see an example in Section 3.4). A proper way of treating this integral may be to extend it to a compactification of $\mathcal{M}_{g,N+2n}$. Note that the first N points $z_1, \dots, z_N$ correspond to the positions of the operators, while the additional $2n$ points $w_1^\pm, \dots, w_n^\pm$ correspond to parameters of the space of maps $\Sigma \to \mathbb{P}^1$. Therefore it is natural to expect that the resulting compactification is related to the Kontsevich moduli space of stable maps.

It follows from that we can write the correlation function $\langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle_{\mathbb{P}^1}$ as

$$\langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle_{\mathbb{P}^1}$$

$$= \langle \mathcal{O}_1 \dots \mathcal{O}_N \exp\left(q^{1/2} \int_{\Sigma} (\Psi_+(w)\pi(w)\overline{\pi}(\overline{w}) + \Psi_-(w)\pi(w)\overline{\pi}(\overline{w})) dw d\overline{w} \right) \rangle_{\mathbb{C}^\times}.$$

Therefore we have interpreted the correlation functions of the $\mathbb{P}^1$ sigma model in the infinite volume as the correlation functions of the deformation of the toric sigma model, with the deformed action

$$(3.3) \quad \frac{i}{2\pi} \int_{\Sigma} d^2z \ (p\partial_{\overline{z}} X + \pi \partial_{\overline{z}} \psi + \overline{p} \partial_z \overline{X} + \overline{\pi} \partial_z \overline{\psi})$$

$$+ q^{1/2} \int_{\Sigma} (\Psi_+(w)\pi(w)\overline{\pi}(\overline{w}) + \Psi_-(w)\pi(w)\overline{\pi}(\overline{w})) d^2z.$$

It is in this sense that we can say that the model with the deformed action (3.3) is equivalent to the type A twisted sigma model with the target $\mathbb{P}^1$ in the infinite volume.

This works fine when Σ has genus zero. But for Σ of genus greater than zero, as we discussed in Section 1.2, we need to take care of the zero modes of p and $\overline{p}$. As we saw above, the existence of these zero modes leads to correlation functions being delta-like differential forms on the moduli spaces of pointed curves. We can regularize these forms by adding the term $\epsilon \int_{\Sigma} p\overline{p} d^2z$ to the action. Note that we are not adding the term corresponding to the inverse of the Fubini-Study form on the target $\mathbb{P}^1$, which would have violated conformal invariance of the action, but rather the inverse of the *flat* metric on $\mathbb{C}^\times$. While this flat metric has poles at $0, \infty \in \mathbb{P}^1$, its inverse has zeroes, and so it is regular on $\mathbb{P}^1$. This term preserves conformal invariance of our theory. There is a similar regularization procedure in the case of more general Fano toric varieties.

This regularization procedure becomes particularly important for maps of low degrees, where without regularization it may be impossible to evaluate the correlation functions.

To illustrate this point, consider the simplest example. Suppose that Σ is the torus and we wish to compute a contribution to some correlation function corresponding to maps of degree one to $\mathbb{P}^1$. While there are certainly no maps from a smooth curve of genus one to $\mathbb{P}^1$, there are stable maps corresponding to curves with nodal singularities having a genus zero component (this is often referred to as "bubbling"). Such maps constitute the entire moduli space of stable maps in this case (unlike the case of maps of high degree, where nodal curves contribute points at the boundary of the locus corresponding to smooth curves). It is well-known that the two-point function of the local observables $\mathcal{O}_1, \mathcal{O}_2$ corresponding to two-forms ω_1, ω_2 on $\mathbb{P}^1$ such that $\int \omega_i = 1$ is equal to $2q$ in this case. If we were to follow the above recipe for the computation of the two-point function in our deformed model literally, we would have to compute a correlation function of the form

$$q\langle \mathcal{O}_1(z_1, \overline{z}_1)\mathcal{O}_2(z_2, \overline{z}_2) \int_\Sigma \Psi_+^{(2)}(w^+, \overline{w}^+)dw^+ d\overline{w}^+ \int_\Sigma \Psi_-^{(2)}(w^-, \overline{w}^-)dw^- d\overline{w}^-\rangle_{\mathbb{C}^\times}.$$

But as we explained above, the integral will be over those points w^+ and w^- which satisfy the Abel-Jacobi condition, which in this case reads $w^+ = w^-$. Since

$$\Psi_+^{(2)}(w^+, \overline{w}^+)\Psi_-^{(2)}(w^-, \overline{w}^-) \to 0$$

as $w^+ \to w^-$, it seems that we obtain 0.

However, if we deform the action by the term $\epsilon \int_\Sigma p\overline{p}d^2z$, the Abel-Jacobi condition is relaxed, and we obtain a non-trivial integral. We will show elsewhere that this integral reproduces the right answer $2q$ when $\epsilon \to 0$. We hope that this is the mechanism by which we can "reach" the components of the moduli spaces of stable maps which cannot be found in the closure of the locus corresponding to smooth curves.

3.2. **Dual description of the deformed theory.**

> А и Б сидели на трубе.
> А упало, Б пропало.
> Кто остался на трубе? [6]

We have come to the key point of our construction. Let us apply the T–duality of Section 2.3 to the deformed theory defined by the action (3.3). In the dual variables R, U the holomortex operators $\Psi_\pm$ become purely local operators $e^{\pm iU}$ and so the action (3.3) becomes

$$(3.4) \quad \frac{i}{2\pi} \int_\Sigma d^2z \; (\partial_z U \partial_{\overline{z}} R + \partial_{\overline{z}} U \partial_z R + \pi \partial_{\overline{z}} \psi + \overline{\pi} \partial_z \psi) + q^{1/2} \int_\Sigma (e^{iU} + e^{-iU})\pi\overline{\pi}d^2z.$$

As we explained in the Introduction, this action is very similar to the action (0.4) of the B twisted Landau-Ginzburg model with the superpotential $W(Y) = q^{1/2}(e^{iY} + e^{-iY})$.

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[6]*A and B were sitting on a pipe. A fell, B disappeared. Who remained on the pipe?* (Russian folklore riddle) The answer is "and", which is "i" in Russian; hence the name "I–model".

Unlike the Lagrangian in (3.3), the Lagrangian in (3.4) is local. The equivalence of the two theories implies that the q–series expansion of the instanton contributions on the deformed model described by (3.3), such as one given by formula (3.2), now has *non-perturbative* meaning in the dual theory defined by (3.4). In this theory $q^{1/2}$ appears as the coupling constant, and if it is small, then expanding the correlation functions in $q^{1/2}$ we reproduce the q–expansion of the correlation functions of the sigma model. However, we can study the theory with the action (3.4) for arbitrary values of $q^{1/2}$.

Note that in the path integral definition of the correlation functions of this model we must integrate over the single-valued function U_0 as well as over the integers $m_j = a_j/2\pi$ appearing in formula (2.10). This leads to some non-trivial consequences. In particular, when Σ has genus greater than zero, the correlation functions involving the factor

$$\prod_{j=1}^{n} e^{-iU(w_j^+)} \prod_{j=1}^{n} e^{iU(w_j^-)}$$

are non-zero only if the divisor $\sum_j (w_j^+) - \sum_j (w_j^-)$ is in the kernel of the Abel-Jacobi map. This follows in the same way as for the toric sigma model (see Section 3.1).

Thus, the action (3.4) defines an intermediate model, which we call the I–*model*, between the A–model, namely, the twisted sigma model with the target $\mathbb{P}^1$ in the infinite volume, and the B–model, namely, the twisted Landau-Ginzburg model with the action (0.4).

By the T–duality of Section 2.3, the q–perturbative I–model is equivalent to the A–model as a conformal field theory. On the other hand, the correlation functions in the BPS sector of the I–model are related to the correlation functions in the BPS sector of the B–model, which is the Landau-Ginzburg model with the superpotential W, considered in [18], up to *contact terms* (in the sense discussed in [24]). Thus, we conclude that the correlation functions in the BPS sector of the A–model are related to the correlation functions in the BPS sector of the B–model Landau-Ginzburg model, up to contact terms. This is usually considered as the statement of mirror symmetry.

Mathematically, this is expressed as the equality of certain generating functions of Gromov-Witten invariants of M (these corresponding to correlation functions in the sigma model deformed by the gravitational descendants) and certain oscillating integrals (these correspond to the correlation functions in a Landau-Ginzburg model). In general, this equivalence involves an intricate transformation on the space of coupling constants that is referred to as the mirror map (see [16], the recent book [19] and references therein for details). The reason for this transformation is that the two theories differ by contact terms, and this difference has to be absorbed in a transformation of the coupling constants (see [24]).

To summarize, our construction for $M = \mathbb{P}^1$ (and for the more general case of a Fano toric variety M treated in the next section) realizes this correspondence of BPS correlation functions in two steps. First, we have an equivalence of two conformal field theories, the twisted sigma model of M (A–model) and the intermediate model defined by the action (3.4) (I–model). This means that all correlation functions that one can write in the A–model and the I–model are equal to each other. Second, we have a correspondence between the I–model to the B–model, which is more subtle: it applies

only to the BPS sector, and in the BPS sector the two models are equivalent only up to contact terms, which is the reason for non-triviality of the mirror map. We do not address here the issue of computing these contact terms and explicitly deriving the mirror map from our proposed equivalence. But in principle this can be done. We hope to return to this issue in a future paper.

3.3. **The supersymmetry charges.** Recall that in the toric sigma model the left and right moving supersymmetry charges are given by the formulas

$$Q = -i \int \psi(z)p(z)dz, \qquad \overline{Q} = -i \int \overline{\psi}(\overline{z})\overline{p}(\overline{z})d\overline{z}.$$

The total supersymmetry charge $Q + \overline{Q}$ corresponds to the de Rham differential, which is typical for an A–model.

After the deformation to the theory with the action (3.3) the supercharges change their form. This is due to the fact that the field $Q(z) = \psi(z)p(z)$ is no longer holomorphic and the field $\overline{Q}(\overline{z})$ is no longer anti-holomorphic in the deformed theory. In fact, for any chiral field $A(z)$ in a conformal field theory, after deforming the action with the term $\int \Phi(z, \overline{z})dzd\overline{z}$, we have the following formula (see [35]):

$$(3.5) \qquad (\partial_{\overline{z}}A)(z, \overline{z}) = \int \Phi(w, \overline{z})dw \cdot A(z),$$

where the integral is over a small contour enclosing z. There is a similar formula for an anti-chiral field.

Suppose that we have a superconformal field theory such that $\Phi = \Psi^{(2)} = G_{-1}\overline{G}_{-1}\Psi$, where Ψ is even, Q–closed and a highest weight vector of the Virasoro algebra, i.e., $L_n\Psi = \overline{L}_n\Psi = 0, n \geq 0$. Let $\Psi^{(1)}$ be the one-form obtained by cohomological descent (see Section 1.4):

$$\Psi^{(1)} = \Psi^{(1)}_z dz + \Psi^{(1)}_{\overline{z}}d\overline{z} = G_{-1}\Psi dz + \overline{G}_{-1}\Psi d\overline{z}.$$

Then if $A(z) = Q(z)$ we find that

$$\int \Psi^{(2)}dw \cdot Q(z) = -\partial_z \Psi^{(1)}_{\overline{z}}.$$

Hence the new left moving supercharge is $\int(Qdz - \Psi^{(1)}_{\overline{z}}d\overline{z})$. Likewise, we have

$$\int \Psi^{(2)}d\overline{w} \cdot \overline{Q}(\overline{z}) = -\partial_{\overline{z}}\Psi^{(1)}_z,$$

and so the new right moving supercharge is $\int(\overline{Q}d\overline{z} - \Psi^{(1)}_z dz)$. Thus, the total supercharge of the deformed theory is

$$Q + \overline{Q} - \int \Psi^{(1)}.$$

In our case we have a deformation by

$$\int q^{1/2}\left(\Psi^{(2)}_+ + \Psi^{(2)}_-\right)dzd\overline{z},$$

where $\Psi_\pm^{(2)} = e^{\mp i \int P} \pi \overline{\pi}$. Therefore we find that

$$\Psi_{\pm,\overline{z}}^{(1)} = \pm i e^{\mp i \int P} \overline{\pi}.$$

Thus, the new supercharge is

$$Q(q) = -i \int \left(\psi p dz - q^{1/2} \left(e^{i \int P} - e^{-i \int P} \right) \overline{\pi} d\overline{z} \right).$$

Similarly, we obtain that after the deformation the supercharge $\overline{Q}$ becomes

$$\overline{Q}(q) = -i \int \left(\overline{\psi} \overline{p} d\overline{z} + q^{1/2} \int \left(e^{i \int P} - e^{-i \int P} \right) \pi dz \right).$$

In the I–model, these supercharges look as follows:

$$Q(q) = -i \int \left(\psi \partial_z U dz - q^{1/2} \left(e^{iU} - e^{-iU} \right) \overline{\pi} d\overline{z} \right),$$

$$\overline{Q}(q) = -i \int \left(\overline{\psi} \partial_{\overline{z}} U d\overline{z} + q^{1/2} \left(e^{iU} - e^{-iU} \right) \pi dz \right),$$

Let us compute the *cohomology of the right moving* supercharge $\overline{Q}(q)$ on the Hilbert space of our theory. This Hilbert space is defined in Section 5.2. We will show in Section 5.4 that the cohomology of the resulting complex coincides with the the cohomology of a complex considered by L. Borisov [4] and F. Malikov and V. Schechtman in [25]. Its cohomology was shown in [25] to be equal to the *quantum cohomology* of $\mathbb{P}^1$. The corresponding cohomology classes may be represented by 1 and $e^{iU} + e^{-iU}$.

On the other hand, according to [34, 20], the cohomology of the operator $\overline{Q}(q)$ in the *perturbative* regime (without instanton corrections) should coincide with the cohomology of the chiral de Rham complex of $\mathbb{P}^1$. To obtain this result, we need to consider a certain degeneration of the above complex, which corresponds to the perturbative regime of the theory. For that we introduce two parameters t_1, t_2 such that $t_1 t_2 = q$, and write $t_1 e^{iU} - t_2 e^{-iU}$ instead of $q^{1/2}(e^{iU} - e^{-iU})$. In the perturbative regime we have $t_1, t_2 \neq 0$, but their product, which is q, becomes equal to 0. In other words, we should work over $\mathbb{C}[t_1, t_2]/(t_1 t_2)$. This corresponds to allowing only degree zero maps $\Sigma \to \mathbb{P}^1$. Such maps can pass through 0 or ∞, but not through both of them.

We will show in Section 5.4 that the cohomology of the degenerate complex coincides with the cohomology of a complex introduced in [4] (see also [25]). Borisov showed in [4] that its cohomology is precisely the cohomology of the chiral de Rham complex of $\mathbb{P}^1$. Therefore we find an agreement with the prediction of [34, 20]. Our computation explains the meaning of the somewhat mysterious computation of [4, 25] from the point of view of the sigma model, with and without instanton corrections.

3.4. **A sample computation of correlation functions.** Here we show how to reproduce the simplest one-instanton calculation of the A–model (the sigma model with the target $\mathbb{P}^1$) in the framework of the I–model defined by action (3.4).

Let $\omega_i, i = 1, 2, 3$, be three two-forms on $\mathbb{P}^1$ representing the second cohomology class. We will assume that they are invariant under the $U(1)$–action on $\mathbb{P}^1$ with the fixed points 0 and ∞. We identify $\mathbb{P}^1 \backslash \{0, \infty\}$ with $\mathbb{C}/2\pi i \mathbb{Z}$ via the exponential map

and use the coordinates R and ϕ on $\mathbb{C}/2\pi i\mathbb{Z} = \mathbb{R} \times i(\mathbb{R}/2\pi\mathbb{Z})$ as before. With respect to these coordinates, these forms may be written as $\omega_i = f_i(R)dX d\overline{X}$, where $X = R + i\phi$. The local operators corresponding to the two-forms ω_i in the A–model are

$$\widehat{\omega}_i = f_i(R)\psi\overline{\psi}.$$

Consider the case when the worldsheet Σ has genus zero. The simplest non-trivial correlation function in the A–model is

$$\langle \widehat{\omega}_1 \widehat{\omega}_2 \widehat{\omega}_3 \rangle_{\mathbb{P}^1} = q \prod_{i=1}^{3} \int_{\mathbb{P}^1} \omega_i.$$

Let us show how to reproduce this answer in the I–model.

In the I–model the operators $\widehat{\omega}_i$ are given by the same formula as above (since R makes sense in the dual theory), hence their correlation function expanded in powers of q is the correlation of the free field theory defined by the action (2.12) given by the formula

$$(3.6) \quad \langle \widehat{\omega}_1 \widehat{\omega}_2 \widehat{\omega}_3 \exp\left(q^{1/2} \int (e^{iU} + e^{-iU})\pi\overline{\pi} \right) \rangle$$

$$= \sum_{n=0}^{\infty} \frac{q^n}{(n!)^2} \langle \widehat{\omega}_1 \widehat{\omega}_2 \widehat{\omega}_3 \left(\int e^{iU}\pi\overline{\pi} \right)^n \left(\int e^{-iU}\pi\overline{\pi} \right)^n \rangle.$$

We have already explained above that, due to the charge conservation, for the correlation function to be non-zero the number of insertions of e^{iU} has to be equal to the number of insertions of e^{-iU}. This explains why in the above formula we consider only the contributions corresponding to equal numbers of insertions.

Next, we count the ghost number. The chiral ghost number of each of the operators $\widehat{\omega}_i$ is one, due to the presence of the fermion ψ. Hence the contribution of the operators $\widehat{\omega}_i$ to the chiral ghost number is 3, and likewise for the anti-chiral ghost number. The conservation law in genus zero is that the total chiral number and the anti-chiral ghost number should be equal to 1. Hence to get a non-zero correlation function we must insert two chiral fermions π and two anti-chiral fermions $\overline{\pi}$. This means that the only non-zero term in the sum (3.6) is the term with $n = 1$, and the coefficient in front of it is precisely q.

Thus, it remains to show that in the free field theory with the action (2.12) we have

$$(3.7) \qquad \langle \prod_{i=1}^{3} \widehat{\omega}_i(z_i, \overline{z}_i) \int e^{iU}\pi\overline{\pi}dw^- d\overline{w}^- \int e^{-iU}\pi\overline{\pi}dw^+ d\overline{w}^+ \rangle = \prod_{i=1}^{3} \int_{\mathbb{P}^1} \omega_i.$$

In the correlation function appearing in the left hand side of this formula we have fixed the points z_1, z_2, z_3 and we are integrating over the points w^- and w^+ the $(1, 1)$–forms $G_{-1}\overline{G}_{-1} \cdot e^{\pm iU}$. By using the Ward identities in the standard way (see [33, 36]), we can "swap" the operators $G_{-1}\overline{G}_{-1}$ and the integrals from the variables w^- and w^+ to any two of the three variables z_1, z_2, z_3, say, z_1 and z_2, fix the position of the remaining point z_3, say $z_3 = \infty$, and fix the positions of w^-, w^+. We find that

$$G_{-1}\overline{G}_{-1} \cdot \widehat{\omega}_i = f_i(R)\partial_z R \partial_{\overline{z}} R dz d\overline{z}.$$

The fermionic part of the correlation function becomes equal to 1, and the bosonic part is given by the integral

$$(3.8) \quad \int d^2X d^2z_1 d^2z_2 \langle f_1(R(z_1, \overline{z}_1)) f_2(R(z_2, \overline{z}_2)) f_3(R(\overline{z}_3))$$

$$\times \, \partial_{z_1} R(z_1) \partial_{\overline{z}_1} R(\overline{z}_1) \partial_{z_2} R(z_2) \partial_{\overline{z}_2} R(\overline{z}_2) e^{iU(w^-)} e^{-iU(w^+)} \rangle$$

(the integral over d^2X is the integral over the zero mode). But we have the following OPE:

$$R(z, \overline{z}) e^{\mp iU(w^\pm)} \sim \pm \log|z - w^\pm| e^{\mp iU(w^\pm)}.$$

Hence

$$\partial_z R(z) \partial_{\overline{z}} R(\overline{z}) e^{\mp iU(w^\pm)} \sim |z - w^\pm|^{\pm 2} e^{\mp iU(w^\pm)}.$$

Therefore the term $\partial_z R(z) \partial_{\overline{z}} R(\overline{z})$ in the correlation function (3.8) may be replaced by $|(z - w^+)/(z - w^-)|$, which is the Jacobian of the map $z \mapsto \log c(z - w^+)/(z - w^-)$. Thus, the integrals over z_1 and z_2 correspond to the integrals of ω_1 and ω_2 over $\mathbb{P}^1$, while the integral over the zero mode corresponds to the integral of ω_3. We find that the integral (3.8) is equal to the right hand side of (3.7), as desired. Note that in this computation we have in effect "localized" on the holomorphic maps $\Sigma \to \mathbb{P}^1$ corresponding to the meromorphic functions $c(z - w^+)/(z - w^-)$, where c is a scalar.

4. General toric varieties

4.1. **Recollections on toric varieties.** Let us recall the combinatorial data involved in the definition of smooth compact toric varieties, following [1] (see also [28]).

Let Λ be a lattice of rank d and $\check{\Lambda}$ be the dual lattice. We set $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, $\Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. For $k \geq 1$ a convex subset $\sigma \subset \Lambda_{\mathbb{R}}$ is called a regular k–dimensional cone if it is generated by a subset of a basis of Λ, i.e.,

$$\sigma = \mathbb{R}_{\geq 0} \langle v_i \rangle_{i=1}^k = \left\{ \sum_{i=1}^k a_i v_i \, \big| \, a_i \in \mathbb{R}_{\geq 0} \right\},$$

where $\{v_1, \ldots, v_k\}$ is a subset of Λ that can be extended to a basis. The 0–dimensional regular cone is by definition the origin $0 \in \Lambda_{\mathbb{R}}$. A subcone σ' of σ generated by a subset of $\{v_i\}_{i=1}^k$ is called a face of σ. In this case we use the notation $\sigma' < \sigma$.

A finite collection $S = \{\sigma_i\}_{i=1}^m$ is called a complete regular fan if the following conditions are satisfied:

(1) if $\sigma \in S$ and $\sigma' < \sigma$, then $\sigma' \in S$;
(2) if $\sigma, \sigma' \in S$, then $\sigma \cap \sigma' < \sigma$ and $\sigma \cap \sigma' < \sigma'$.
(3) $\Lambda_{\mathbb{R}} = \bigcup_{i=1}^m \sigma_i$.

For example, let Λ be the d–dimensional lattice generated by $v_1, \ldots, v_d$. Set $v_{d+1} = -\sum_{i=1}^d v_i$. For any subset $I \subset \{1, \ldots, d+1\}$, let $\sigma_I = \mathbb{R}_{\geq 0} \langle v_j \rangle_{j \in I}$. Then $S(d) = \{\sigma_I\}_{I \subset \{1, \ldots, d+1\}}$ is a complete regular fan.

One associates a toric variety to a fan S as follows. To each cone $\sigma \in S$ we assign the dual cone in $\check{\Lambda}$,

$$\check{\sigma} = \{\check{\lambda} \in \check{\Lambda} \, | \, \langle \check{\lambda}, v \rangle \geq 0, \forall v \in \sigma\},$$

and the affine variety $\mathbb{A}_\sigma = \operatorname{Spec} \mathbb{C}[\check{\sigma}]$. It is clear that if $\sigma' < \sigma$, then we have a natural inclusion $\mathbb{A}_\sigma \hookrightarrow \mathbb{A}_{\sigma'}$. This allows us to glue the varieties $\mathbb{A}_\sigma, \sigma \in S$, into a projective variety $\mathbb{P}_S$, which is the toric variety associated to S.

For example, the variety associated to the fan $S(d)$ is the projective variety $\mathbb{P}^d$.

In particular, we have an open dense subvariety of $\mathbb{P}_S$,

$$\mathbb{T}_S = \mathbb{A}_{\{0\}} = \operatorname{Spec} \mathbb{C}[\check{\Lambda}] \simeq \operatorname{Spec} \mathbb{C}[x_i^{\pm 1}]_{i=1}^d = (\mathbb{C}^\times)^d.$$

Here $x_i, i = 1, \ldots, d$, are coordinates on $\mathbb{T}_S$ corresponding to a basis $\{\check{e}_1, \ldots, \check{e}_d\}$ of $\check{\Lambda}$ that is dual to a basis $\{e_1, \ldots, e_d\}$ of Λ that we fix once and for all. Note that any element $\check{\lambda} = \sum_{i=1}^d a_i \check{e}_i$ gives rise to a monomial function $\prod_{i=1}^d x_i^{a_i}$ on $\mathbb{T}_S$ which we denote by $f_{\check{\lambda}}$.

In a basis independent way we can say that $\mathbb{T}_S$ is the algebraic torus, whose lattices of characters $\mathbb{T}_S \to \mathbb{C}^\times$ and cocharacters $\mathbb{C}^\times \to \mathbb{T}_S$ are canonically identified with $\check{\Lambda}$ and Λ, respectively.

Let $\sigma(1), \ldots, \sigma(N)$ be the set of all one-dimensional cones in S. Each such cone $\sigma(i)$ has a canonical generator $v(i) \in \Lambda$ that can be completed to a basis of Λ. The varieties $\mathbb{A}_{\sigma(i)}, i = 1, \ldots, N$ provide a covering of the toric variety $\mathbb{P}_S$ by open dense subsets. By definition, the ring of functions on $\mathbb{A}_{\sigma(i)}$ is the span of all monomials $f_{\check{\lambda}}$, where $\langle \check{\lambda}, v(i) \rangle \geq 0$. The complement of $\mathbb{T}_S$ in $\mathbb{A}_{\sigma(i)}$ is the divisor C_i in the latter whose ideal is the span of the monomials $f_{\check{\lambda}}$, where $\langle \check{\lambda}, v(i) \rangle > 0$. It is clear that the closures $\overline{C}_i$ of these divisors are the irreducible components of the complement of $\mathbb{T}_S$ in $\mathbb{P}_S$.

For instance, in the case of $\mathbb{P}^d$, the one-dimensional cones are $\sigma(i) = \mathbb{R}_{\geq 0} v_i, i = 1, \ldots, d+1$, and so $v(i) = v_i$. Therefore the varieties $\mathbb{A}_{\sigma(i)}$ are the subvarieties of $\mathbb{P}^d$, where all but the ith homogeneous components are non-zero. The divisor C_i consists of points in which the ith homogeneous component is equal to 0.

4.2. **The toric sigma model.** Let us fix a smooth compact toric variety $\mathbb{P}_S$ corresponding to a fan S. We will assume that $\mathbb{P}_S$ is a fano variety. In fact, our construction can be applied to more general toric varieties; however, in the case of toric varieties that are not Fano the connection between the deformed model that we define below and the A–model of $\mathbb{P}_S$ is more subtle. We have indicated some of the underlying reasons for this in Section 3.1.

The first step of our construction is to define the toric sigma model with the target

$$\mathbb{T}_S \simeq (\mathbb{C}^\times)^d = \operatorname{Spec} \mathbb{C}[x_i^{\pm 1}]_{i=1}^d.$$

This model is just the tensor product of d independent copies of the toric sigma model of $C^\times$ described in Section 2.1. We will use the logarithmic coordinates $X_i, i = 1, \ldots, d$, on $(\mathbb{C}^\times)^d \simeq \Lambda_\mathbb{C}/2\pi i \Lambda$, such that $x_i = e^{X_i}$. Thus, we have the fields X^i, p_i, ψ^i, π_i and their complex conjugates $X^{\bar{i}}, p_{\bar{i}}, \psi^{\bar{i}}, \pi_{\bar{i}}$.

For any element $\check{\lambda} = \sum_{i=1}^d a_i \check{e}_i \in \check{\Lambda}$ we have fields $X^{\check{\lambda}} = \sum_{i=1}^d a_i X^i$ and $\overline{X}^{\check{\lambda}} = \sum_{i=1}^d a_i X^{\bar{i}}$, whereas for any element $\lambda = \sum_{i=1}^d b_i e_i \in \Lambda$ we have fields $p_\lambda = \sum_{i=1}^d b_i p_i$ and $\overline{p}_\lambda = \sum_{i=1}^d b_i p_{\bar{i}}$. We define the fermions $\psi^{\check{\lambda}}, \overline{\psi}^{\check{\lambda}}, \check{\lambda} \in \check{\Lambda}$, and $\pi_\lambda, \overline{\pi}_\lambda, \lambda \in \Lambda$ in the same way.

The action of the toric sigma model is given by the formula

$$(4.1) \qquad \frac{i}{2\pi} \int_\Sigma d^2z \, \left(p_i \partial_{\overline{z}} X^i + \pi_i \partial_{\overline{z}} \psi^i + p_{\overline{i}} \partial_z X^{\overline{i}} + \pi_{\overline{i}} \partial_z \psi^{\overline{i}} \right).$$

The theory has $N = (2, 2)$ superconformal symmetry. The corresponding generators are the sums of the generators in the $\mathbb{C}^\times$ toric sigma model given by formula (2.6).

As in the one-dimensional case, explained in Section 2.2, we find that the fields X^i may have non-trivial winding. The winding numbers take values in the lattice Λ. For each $\lambda \in \Lambda$ we introduce the corresponding *holomortex operators*

$$\Psi_\lambda(z, \overline{z}) = e^{-i \int P_\lambda} = \exp\left(-i \int_{z_0}^z (p_\lambda(w) dw + \overline{p}_\lambda(\overline{w}) d\overline{w}) \right).$$

They have the following OPE with the fields $X^{\check{\mu}}$ and $\overline{X}^{\check{\mu}}$:

$$X^{\check{\mu}}(z) \Psi_\lambda(w, \overline{w}) = \langle \check{\mu}, \lambda \rangle \log(z - w) \Psi_\lambda(w, \overline{w}),$$
$$\overline{X}^{\check{\mu}}(z) \Psi_\lambda(w, \overline{w}) = \langle \check{\mu}, \lambda \rangle \log(\overline{z} - \overline{w}) \Psi_\lambda(w, \overline{w}).$$

The prescription for the computation of correlation functions of these operators is the same as in the one-dimensional case (see Section 2.2).

Next, we define the T–dual theory of the $\mathbb{T}_S$–toric sigma model. This is an ordinary sigma model with the target being the *partially dualized torus*

$$\check{\mathbb{T}}_S = \Lambda_{\mathbb{R}} \times i(\check{\Lambda}_{\mathbb{R}}/2\pi\check{\Lambda}),$$

equipped with the Minkowski metric, which is the product of d copies of the Minkowski metric introduced in Section 2.3. Note that this metric is canonically defined precisely because the the lattices Λ and $\check{\Lambda}$ are dual to each other.

In the dual theory the bosonic fields are U_i and R^i, $i = 1, \ldots, d$, and the fermionic fields are the same as in the toric sigma model. The action is as in (2.12):

$$(4.2) \qquad \widetilde{I} = \frac{i}{2\pi} \int_\Sigma d^2z \, (\partial_z U_j \partial_{\overline{z}} R^j + \partial_{\overline{z}} U_j \partial_z R^j + \pi_j \partial_{\overline{z}} \psi^j + \pi_{\overline{j}} \partial_z \psi^{\overline{j}}).$$

The transformation formulas for the bosonic fields of the two models are

$$p_i(z) = \partial_z U_i(z, \overline{z}), \qquad p_{\overline{i}}(\overline{z}) = \partial_{\overline{z}} U_i(z, \overline{z}),$$

$$\frac{1}{2}(X^i(z) + X^{\overline{i}}(\overline{z})) = R^i(z, \overline{z}).$$

The holomortex operators $\Psi_\lambda = e^{-i \int P_\lambda}$ have a simple realization in the dual variables:

$$\Psi_\lambda(z, \overline{z}) = e^{-iU_\lambda(z, \overline{z})},$$

where we set $U_\lambda = \sum_{i=1}^d b_i U_i$ for $\lambda = \sum_{i=1}^d b_i e_i$.

4.3. **Changing the target from $\mathbb{T}_S$ to $\mathbb{P}_S$.** We wish to describe the non-linear sigma model with the target toric variety $\mathbb{P}_S$ as a deformation of the toric sigma model with the target torus $\mathbb{T}_S$. We follow the same idea as in the case of $\mathbb{P}^1$ explained in Section 3. Recall from Section 4.1 that the complement of $\mathbb{T}_S$ in $\mathbb{P}_S$ is a divisor, whose irreducible components $\overline{C}_j, j = 1, \ldots, N$, are naturally parameterized by the one-dimensional cones σ_j in S generated by $v(j) \in \Lambda$. A generic holomorphic map $\Phi : \Sigma \to \mathbb{P}_S$ takes values in $\mathbb{T}_S \subset \mathbb{P}_S$ for all but finitely many points, and at the special points it takes values in the open part C_j of the divisor $\overline{C}_j$, introduced in Section 4.1, for some $j = 1, \ldots, N$. Let us denote the points of Σ where Φ takes values in C_i by $w_k^{(j)}, j = 1, \ldots, m_j$.

We propose to *include such maps by inserting in the correlation functions the holomortex operators* $\Psi_{v(j)}(w_k^{(j)}, \overline{w}_k^{(j)})$ introduced in the previous section. Recall that

$$\Psi_{v(j)}(w, \overline{w}) = e^{-i \int_{w_0}^{w} P_{v(j)}}.$$

Clearly, these operators are Q–closed. Hence we find the following formula for the two-form cohomological descendant field of $\Psi_{v(j)}(w, \overline{w})$:

$$\Psi_{v(j)}^{(2)}(w, \overline{w}) dw d\overline{w} = \Psi_{v(j)}(w, \overline{w}) \pi_{v(j)}(w) \overline{\pi}_{v(j)}(\overline{w}) dw d\overline{w}.$$

Now observe that the lattice of all relations between the generators $v(j), j = 1, \ldots, N$, of one-dimensional cones in S is generated by $N - d$ linearly independent relations

$$\sum_{j=1}^{N} a_{ij} v(j) = 0, \qquad i = 1, \ldots, N - d,$$

where we choose the a_{ij}'s to be integers that are relatively prime.

Let us introduce parameters $t_j, j = 1, \ldots, N$, and set

$$(4.3) \qquad\qquad q_i = \prod_{j=1}^{N} t_j^{a_{ij}}, \qquad i = 1, \ldots, N - d.$$

As in the case of $\mathbb{P}^1$, the type A twisted sigma model with the target $\mathbb{P}_S$ in the infinite volume is then described by the deformation of the toric sigma model by

$$\sum_{j=1}^{N} t_j \int_{\Sigma} \Psi_{v(j)} \pi_{v(j)} \overline{\pi}_{v(j)} dw d\overline{w}.$$

Note that the t_j's can be redefined by changing the normalization of the operators $\Psi_{v(j)}$, but this will not affect the parameters q_i given by formula (4.3). Therefore the q_i's are the true parameters of the theory, and they correspond precisely to the Kähler classes on $\mathbb{P}_S$, as explained in [1].

For example, if $\mathbb{P}_S = \mathbb{P}^d$, then we have

$$\Psi_{v(j)} \pi_{v(j)} \overline{\pi}_{v(j)} = e^{-i \int P_j} \pi_j \overline{\pi}_j, \qquad j = 1, \ldots, d,$$

$$\Psi_{v(d+1)} \pi_{v(d+1)} \overline{\pi}_{v(d+1)} = e^{i \sum_{j=1}^{d} \int P_j} \left(\sum_{j=1}^{d} \pi_j \right) \left(\sum_{j=1}^{d} \pi_{\overline{j}} \right),$$

and there is only one parameter $q = \prod_{j=1}^{d+1} t_j$.

4.4. **The I–model.** Finally, we apply the T–duality to the action of the deformed toric sigma model. The operators $\Psi_{v(j)}$ are now written as $e^{-iU_{v(j)}}$, and so the action takes the form

$$(4.4) \qquad \frac{i}{2\pi} \int_\Sigma d^2z \left(\partial_z U_j \partial_{\bar{z}} R^j + \partial_{\bar{z}} U_j \partial_z R^j + \pi_j \partial_{\bar{z}} \psi^j + \pi_{\bar{j}} \partial_z \psi^{\bar{j}} \right) + \int_\Sigma \widetilde{W} d^2z,$$

where

$$\widetilde{W} = \sum_{j=1}^N t_j e^{-iU_{v(j)}} \pi_{v(j)} \overline{\pi}_{v(j)}.$$

For example, if $\mathbb{P}_S = \mathbb{P}^d$, then we have

$$\widetilde{W} = \sum_{j=1}^d t_j e^{-iU_j} \pi_j \pi_{\bar{j}} + t_{d+1} e^{i \sum_{j=1}^d U_j} \left(\sum_{j=1}^d \pi_j \right) \left(\sum_{j=1}^d \pi_{\bar{j}} \right).$$

The action (4.4) defines the I–model for a general toric variety $\mathbb{P}_S$. As in the case of $\mathbb{P}^1$, the action (4.4) should be compared to the action of the Landau-Ginzburg model with the superpotential

$$(4.5) \qquad W = \sum_{j=1}^N t_j e^{-iY_{v(j)}}.$$

Here $Y_{v(j)}$ is a chiral superfield which is a linear combination of d independent chiral superfields $Y_k, k = 1, \ldots, d$, defined by the formula $Y_{v(j)} = \sum_{k=1}^d b_{ij} Y_k$, where $v(j) = \sum_{k=1}^d b_{ij} e_k$. We recognize in formula (4.5) the superpotential of the type B twisted Landau-Ginzburg model that is mirror dual to the type A sigma model with the target $\mathbb{P}_S$ considered in [18].

As we explained in the case of $\mathbb{P}^1$, this suggests that mirror symmetry can be realized in two steps. The first step is the equivalence of the twisted sigma model of $\mathbb{P}_S$ (A–model), described as a deformation of a free field theory, and the intermediate model defined by the action (4.4) (I–model), as conformal field theories. The second step is a correspondence between the I–model to the B–model, which is more subtle: it applies only to the BPS sector, and in the BPS sector the two models are equivalent only up to contact terms.

We hope that further study of the I–model and its connections with the A–model on the one hand and the B-model on the other hand will help us understand more fully the nature of mirror symmetry.

4.5. **Supercharges.** Let us compute the supersymmetry charges of the I–model. Following the same computation as in Section 3.3, we find the following formulas for the

left and right moving supercharges:

$$Q = -i \int \left(\psi^k \partial_z U_k dz + \left(\sum_{j=1}^{N} t_j e^{-iU_{v(j)}} \overline{\pi}_{v(j)} d\overline{z} \right) \right),$$

$$\overline{Q} = -i \int \left(\psi^{\overline{k}} \partial_{\overline{z}} U_k d\overline{z} - \left(\sum_{j=1}^{N} t_j e^{-iU_{v(j)}} \pi_{v(j)} dz \right) \right).$$

It is interesting to compute the cohomologies of the right moving supercharge $\overline{Q}(q)$. As in the case of $\mathbb{P}^1$, we will find in Section 5.4 that these cohomologies coincide with the cohomologies of a complex constructed in [4, 25]. It was shown in [25] that in the case of $\mathbb{P}^n$ this cohomology coincides with the quantum cohomology of $\mathbb{P}^n$. We expect the same to be true for more general toric Fano varieties. (In fact, it follows from the results of [25] that the cohomology of the total supercharge $Q + \overline{Q}$ is isomorphic to the quantum cohomology of $\mathbb{P}_S$.) On the other hand, the cohomology of a degeneration of this complex was computed in [4], and it gives the cohomology of the chiral de Rham complex of $\mathbb{P}_S$. This agrees with the prediction of [34, 20].

5. Operator formalism

In this section we discuss the operator content of the toric sigma models introduced in the previous sections, the T–duality transform and the deformed models. For simplicity we will mostly treat the case of the target $\mathbb{C}^\times$ as the general case is very similar. The algebraic object that we define (we call it the "Hilbert space" of the theory) obeys the axioms of a vertex algebra mixing chiral and anti-chiral sectors, similar to the ones defined by A. Kapustin and D. Orlov in [21] who considered the case of sigma models of the torii in the finite volume. In particular, we define a state-field correspondence assigning to every state of the Hilbert space an operator depending on $z, \overline{z}$ acting on the Hilbert space. Thus, the toric sigma models and their T–dual models studied in this paper provide us with new examples of vertex algebras in which chiral and anti-chiral sectors are non-trivially mixed. While there is a vast mathematical literature on the subject of chiral algebras, examples of mixed vertex algebras have not been widely discussed in the mathematical literature so far. Actually, it is expected that the vertex algebras that occur in the study of mirror symmetry are for the most part of this sort, with the chiral and anti-chiral sectors entangled in a non-trivial way. Therefore we believe that algebraic study of such vertex algebras is important.

At the end of this section we will compute the chiral algebra of the I–model and the cohomology of the right moving supercharge, making a connection with the results of [4, 25].

5.1. **Hilbert space and state-field correspondence in the toric sigma model.**
We collect all the ingredients found in Section 2 and define the Hilbert space and the state-field correspondence of the toric sigma model.

Let us write

$$X(z) = \omega \log z + \sum_{n \in \mathbb{Z}} X_n z^{-n},$$

$$\overline{X}(\overline{z}) = \omega \log \overline{z} + \sum_{n \in \mathbb{Z}} \overline{X}_n \overline{z}^{-n},$$

$$p(z) = \sum_{n \in \mathbb{Z}} p_n z^{-n-1},$$

$$p(z) = \sum_{n \in \mathbb{Z}} \overline{p}_n \overline{z}^{-n-1},$$

and let T_m be the operator satisfying

$$[\omega, T_m] = m T_m$$

and commuting with the X_n's and p_n's. Note that we also have the following commutation relations:

$$[X_n, p_m] = -i\delta_{n,-m},$$
$$[p_n, p_m] = [X_n, X_m] = 0,$$

and ω commutes with all p_n's and X_n's. We also have similar formulas for the components of the anti-chiral fields.

Consider the Heisenberg algebras generated by $X_n, p_n, n \in \mathbb{Z}$, and $\overline{X}_n, \overline{p}_n, n \in \mathbb{Z}$, respectively. For $\gamma \in \mathbb{C}$, let $\mathcal{F}_\gamma$ (resp., $\overline{\mathcal{F}}_\gamma$) be the Fock representation of the Heisenberg algebra generated by a vector annihilated by $X_n, n > 0, p_m, m \geq 0$ (resp., $\overline{X}_n, n > 0, \overline{p}_m, m \geq 0$) and on which ip_0 (resp., $i\overline{p}_0$) acts by multiplication by γ. Since the imaginary part of $X(z)$ is periodic, the eigenvalues of $i(p_0 - \overline{p}_0)$ are quantized to be integers. This is exactly the condition that we obtained in Section 2.2. The operator ω is also quantized and has to take integer eigenvalues, called the winding numbers.

The Fock representation $\mathcal{F}_\alpha$ (resp., $\overline{\mathcal{F}}_\alpha$) on which ω acts by multiplication by $m \in \mathbb{Z}$ will be denoted $\mathcal{F}_{\alpha,m}$ (resp., $\overline{\mathcal{F}}_{\alpha,m}$). We denote by $|\alpha, n\rangle$ (resp., $\overline{|\alpha, m\rangle}$) its generating vector.

The big bosonic Hilbert space of the theory is the direct product of the tensor products of the left and right moving Fock representations

$$(5.1) \qquad \mathcal{F}_{(r+\alpha)/2,m} \otimes \overline{\mathcal{F}}_{(-r+\alpha)/2,m},$$

where $r, m \in \mathbb{Z}$ and α runs over a subset of $\mathbb{C}$. There are different choices for this subset which is determined by what type of functions of the zero mode R_0 of the field $R(z, \overline{z}) = (X(z) + \overline{X}(\overline{z}))/2$ we wish to allow.

One possibility is to restrict ourselves to the subset of $\alpha \in i\mathbb{R} \subset \mathbb{C}$. This is compatible with the structure of the Hilbert space in the sigma model at the finite radius $\sqrt{t}$ and corresponds to restricting ourselves to the L_2 functions of the zero mode. This choice is natural from the point of view of the latter model because it can itself be obtained as the sigma model with the target torus of radii $\sqrt{t}$ and r in the limit when $r \to \infty$.

But this is not the only way to treat the toric sigma model. Indeed, we will consider it as a degeneration of the sigma model with the target $\mathbb{P}^1$ (in the infinite volume limit).

Therefore another natural choice for the class of functions of the zero mode is the space of polynomial functions in $e^{\pm R_0}$. In fact, it is natural to consider all rational functions on $\mathbb{P}^1$ which are regular on $\mathbb{C}^\times$, that is polynomials in $e^{\pm X_0}$, as well as their complex conjugates, polynomials in $e^{\pm \overline{X}_0}$. Choosing this space is equivalent to demanding that α be in the set $\mathbb{Z} \subset \mathbb{C}$. In the subsequent sections we will define a deformation of the toric sigma model which is equivalent to the A–model of $\mathbb{P}^1$ (that is the type A twisted sigma model with the target $\mathbb{P}^1$ in the infinite volume). The operators in this theory will be obtained by restriction to $\mathbb{C}^\times$ from operators defined on the entire $\mathbb{P}^1$. While there are no regular functions on $\mathbb{P}^1$ other than constant functions, there will be composite operators depending on $X(z)$ as well as $p(z)$ that are well-defined, such as the normally ordered products $:e^{\pm X(z)}p(z):$.

Thus, we see that there are different choices for the subset of α's which we may include in our Hilbert space. However, from the purely algebraic point of view, the state-field correspondence that we will now describe works equally well for any of these choices. Therefore in the rest of this subsection we will consider the direct product of the Fock spaces (5.1) with arbitrary complex values of α.

The state-field correspondence that we describe now gives us the structure of a vertex algebra combining holomorphic and anti-holomorphic sections, in the sense of [21]. Note that just like in the case of sigma models on the torii that was considered in [21], we cannot separate the holomorphic and anti-holomorphic sectors of this vertex algebra.

The key point is the assignment of a field to the state $|0, m\rangle \otimes \overline{|0, m\rangle}$. We assign to it the field $e^{-im \int P}$ given by the formula

$$(5.2) \quad e^{-im \int P}(z, \overline{z}) = \exp\left(-im \int^z (p(w)dw + \overline{p}(\overline{w})d\overline{w}) \right) \overset{\text{def}}{=}$$

$$T_m |z|^{-im(p_0 + \overline{p}_0)} \left(\frac{z}{\overline{z}} \right)^{-im(p_0 - \overline{p}_0)/2} \exp\left(im \sum_{n \neq 0} \frac{p_n z^{-n} + \overline{p}_n \overline{z}^{-n}}{n} \right),$$

where T_m is the translation operator that shifts the winding number by m and commutes with all other operators. Note that the operator $i(p_0 - \overline{p}_0)$ has only integer eigenvalues on the Hilbert space of the theory, so this formula is well-defined.

This field has the following OPE with $X(z)$:

$$X(z)e^{-im \int P}(w, \overline{w}) = m \log(z - w)e^{-im \int P}(w, \overline{w}),$$

and similarly with $\overline{X}(\overline{z})$.

Next, we define the field corresponding to the state

$$|(r + \alpha)/2, m\rangle \otimes \overline{|(-r + \alpha)/2, m\rangle}$$

as the normally ordered product

$$:e^{(r+\alpha)X(z)/2 + (-r+\alpha)\overline{X}(\overline{z})/2} e^{-im \int P}(z, \overline{z}):,$$

where

$$e^{(r+\alpha)X(z)/2+(-r+\alpha)\overline{X}(\overline{z})/2} \stackrel{\text{def}}{=}$$

$$|z|^{\omega\alpha} \left(\frac{z}{\overline{z}}\right)^{r\omega/2} S_{(r+\alpha)/2,(-r+\alpha)/2} \exp\left(\frac{1}{2}(r+\alpha)\sum_{n\neq 0} X_n z^{-n} + \frac{1}{2}(-r+\alpha)\sum_{n\neq 0} \overline{X}_n \overline{z}^{-n}\right).$$

Here $S_{(r+\alpha)/2,(-r+\alpha)/2}$ is the translation operator

$$\mathcal{F}_{(r'+\alpha')/2,m} \otimes \overline{\mathcal{F}}_{(-r'+\alpha')/2,m} \to \mathcal{F}_{(r+r'+\alpha+\alpha')/2,m} \otimes \overline{\mathcal{F}}_{(-r-r'+\alpha+\alpha')/2,m}.$$

Note that since ω has only integer eigenvalues, this formula is well-defined.

Finally, the fields corresponding to other states in the Fock representation $\mathcal{F}_{r+\alpha,m} \otimes \overline{\mathcal{F}}_{-r+\alpha,m}$ are constructed as the normally ordered products of the fields defined above and the fields $\partial_z X(z), p(z), \partial_{\overline{z}}\overline{X}(\overline{z}), \overline{p}(\overline{z})$, under the usual assignment:

$$X_n \mapsto \frac{1}{(-n)!}\partial_z^{-n} X(z), \quad n \leq 0; \qquad p_n \mapsto \frac{1}{(-n-1)!}\partial_z^{-n-1} p(z), \quad n < 0.$$

This completes the description of the bosonic part of the Hilbert space of the theory.

Now we describe the fermionic part. Let us write

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n}, \qquad \pi(z) = \sum_{n\in\mathbb{Z}} \pi_n z^{-n-1},$$

and similarly for the anti-chiral fields. The operator product expansions give us the following anti-commutation relations:

$$[\pi_n, \psi_m]_+ = -i\delta_{n,-m}, \qquad [\pi_n, \pi_m]_+ = [\psi_n, \psi_m]_+ = 0,$$

and similarly for the components of the anti-chiral fields. Consider the Clifford algebra generated by $\psi_n, \pi_n, n \in \mathbb{Z}$ (resp., $\overline{\psi}_n, \overline{\pi}_n, n \in \mathbb{Z}$) and let $\mathcal{F}_{\text{ferm}}$ (resp., $\overline{\mathcal{F}}_{\text{ferm}}$) be the fermionic Fock space representation of this algebra generated by a vector annihilated by $\psi_n, n > 0, \pi_m, m \geq 0$ (resp., $\overline{\psi}_n, n > 0, \overline{\pi}_m, m \geq 0$). The fermionic Hilbert space is $\mathcal{F}_{\text{ferm}} \otimes \overline{\mathcal{F}}_{\text{ferm}}$. The total Hilbert space of the theory is the tensor product of the bosonic and fermionic spaces.

5.2. T–**duality, operator formalism.** Next, we discuss the duality transformation from the operator point of view. The operator content of the toric sigma model is described in the previous section. Let is now describe the operator content of the T–dual free bosonic field theory given by the action (2.11). The equations of motion imply that the fields U and R are harmonic, so we can write

$$R(z,\overline{z}) = R_0 + \log|z|p_R + \sum_{n\neq 0}\frac{R_n}{n}z^{-n} + \sum_{n\neq 0}\frac{\overline{R}_n}{n}\overline{z}^{-n},$$

$$U(z,\overline{z}) = U_0 + \log|z|p_U - \frac{i}{2}\widetilde{\omega}\log\frac{z}{\overline{z}} + \sum_{n\neq 0}\frac{U_n}{n}z^{-n} + \sum_{n\neq 0}\frac{\overline{U}_n}{n}\overline{z}^{-n}.$$

The OPEs of these fields are of the form

$$R(z,\overline{z})U(w,\overline{w}) = i\log|z-w| + :R(z,\overline{z})U(w,\overline{w}):.$$

The Fourier coefficients of these fields satisfy the following commutation relations

$$[R_n, U_m] = \frac{i}{2} n \delta_{n,-m}, \qquad [R_n, R_m] = [U_n, U_m] = 0,$$

$$[\overline{R}_n, \overline{U}_m] = \frac{i}{2} n \delta_{n,-m}, \qquad [\overline{R}_n, \overline{R}_m] = [\overline{U}_n, \overline{U}_m] = 0,$$

and

$$[R_0, p_U] = -i, \qquad [U_0, p_R] = i.$$

All other commutators are equal to zero. Because U is periodic, the momentum operator p_R is quantized and takes only integer values, whereas p_U can take arbitrary values. Also, the winding operator $\widetilde{\omega}$ is quantized and takes only integer values.

The Hilbert space of the theory is built from Fock representations of the Heisenberg algebra generated by the coefficients in the expansions of R and U. For $\beta \in \mathbb{C}$ and $r, m \in \mathbb{Z}$, let $\widetilde{\mathfrak{F}}_{r,\beta,m}$ be the Fock representation generated by a vector $|r, \beta, m\rangle$ annihilated by $R_n, U_n, \overline{R}_n, \overline{U}_n, n > 0$, and on which p_R and p_U act by multiplication by m and by β, respectively, and $\widetilde{\omega}$ acts by multiplication by r.

Introduce the following translation operators, which map generating vectors to generating vectors and commute with the operators $R_n, U_n, \overline{R}_n, \overline{U}_n, n \neq 0$:

$$e^{\beta R_0} : \qquad \widetilde{\mathfrak{F}}_{r',\beta',m'} \to \widetilde{\mathfrak{F}}_{r',\beta'+\beta,m'},$$

$$e^{imU_0} : \qquad \widetilde{\mathfrak{F}}_{r',\beta',m'} \to \widetilde{\mathfrak{F}}_{r',\beta,m'+m},$$

$$e^{r\widehat{R}_0} : \qquad \widetilde{\mathfrak{F}}_{r',\beta',m'} \to \widetilde{\mathfrak{F}}_{r'+r,\beta',m'}.$$

The state-field correspondence is defined as follows. The field corresponding to the vector $|r, \beta, m\rangle$ is given by the normally ordered product

(5.3) $:e^{\beta R(z,\overline{z})} e^{imU(z,\overline{z})} e^{r\widehat{R}(z,\overline{z})}:$,

where

$$e^{\beta R(z,\overline{z})} = e^{\beta R_0} |z|^{\beta p_R} : \exp\left(\beta \sum_{n \neq 0} \frac{R_n}{n} z^{-n} + \beta \sum_{n \neq 0} \frac{\overline{R}_n}{n} \overline{z}^{-n} \right) :,$$

$$e^{imU(z,\overline{z})} = e^{imU_0} |z|^{imp_U} \left(\frac{z}{\overline{z}} \right)^{m\widetilde{\omega}/2} : \exp\left(im \sum_{n \neq 0} \frac{U_n}{n} z^{-n} + m \sum_{n \neq 0} \frac{\overline{U}_n}{n} \overline{z}^{-n} \right) :,$$

$$e^{r\widehat{R}(z,\overline{z})} = e^{r\widehat{R}_0} \left(\frac{z}{\overline{z}} \right)^{rp_R/2} : \exp\left(r \sum_{n \neq 0} \frac{R_n}{n} z^{-n} - r \sum_{n \neq 0} \frac{\overline{R}_n}{n} \overline{z}^{-n} \right) :.$$

The other fields are obtained in the standard way as normally ordered products of the field (5.3) and the derivatives of R and U, under the rule

$$R_n \mapsto \frac{1}{(-n-1)!} \partial_z^{-n} R(z,\overline{z}), \qquad \overline{R}_n \mapsto \frac{1}{(-n-1)!} \partial_{\overline{z}}^{-n} R(z,\overline{z}),$$

$$U_n \mapsto \frac{1}{(-n-1)!} \partial_z^{-n} U(z,\overline{z}), \qquad \overline{U}_n \mapsto \frac{1}{(-n-1)!} \partial_{\overline{z}}^{-n} U(z,\overline{z}),$$

for $n < 0$.

The isomorphism between the Hilbert spaces of the two theories is given by the following transformation of the generating fields:

$$(5.4) \qquad \frac{1}{2}(X(z) + \overline{X}(\overline{z})) \mapsto R(z, \overline{z}),$$

$$(5.5) \qquad p(z) \mapsto \partial_z U(z, \overline{z}), \qquad \overline{p}(\overline{z}) \mapsto \partial_{\overline{z}} U(z, \overline{z}),$$

$$(5.6) \qquad \frac{1}{2}(X(z) - \overline{X}(\overline{z})) \mapsto \widehat{R}(z, \overline{z}).$$

The field $\widehat{R}(z, \overline{z})$ is non-local with respect to $R(z, \overline{z})$, namely, $\widehat{R}(z, \overline{z}) = R_-(z) - R_+(\overline{z})$, where $R_\pm$ are the holomorphic and anti-holomorphic parts of $R(z, \overline{z}) = R_-(z) + R_+(\overline{z})$ defined by the formulas

$$(5.7) \qquad R_-(z) = \frac{1}{2}(R_0^- + p_R \log z) + \sum_{n \neq 0} \frac{R_n}{n} z^{-n},$$

$$R_+(\overline{z}) = \frac{1}{2}(R_0^+ + p_R \log \overline{z}) + \sum_{n \neq 0} \frac{\overline{R}_n}{n} \overline{z}^{-n},$$

where $R_0^\pm = R_0 \mp \widehat{R}_0$.

More precisely, at the level of the operators appearing as the coefficients in the expansions of these fields we have the following transformation:

$$X_n \mapsto \frac{2}{n} R_n, \qquad \overline{X}_n \mapsto \frac{2}{n} \overline{R}_n, \qquad n \neq 0,$$

$$p_n \mapsto -U_n, \qquad \overline{p}_n \mapsto -\overline{U}_n, \qquad n \neq 0,$$

$$\frac{1}{2}(X_0 + \overline{X}_0) \mapsto R_0, \qquad \frac{1}{2}(X_0 - \overline{X}_0) \mapsto \widehat{R}_0,$$

$$(p_0 + \overline{p}_0) \mapsto p_U,$$

$$i(p_0 - \overline{p}_0) \mapsto \widetilde{\omega}, \qquad \omega \mapsto p_R.$$

Thus we see that this transformation exchanges the momentum and the winding, as expected in T–duality. The isomorphism between the two Hilbert spaces is given by sending $\mathcal{F}_{r+\alpha,m} \otimes \overline{\mathcal{F}}_{-r+\alpha,m}$ to $\widetilde{\mathcal{F}}_{r,\alpha,m}$.

The fermionic Hilbert spaces are the same in the two theories. Hence we obtain an isomorphism of the full Hilbert spaces of the two T–dual theories.

5.3. **Chiral algebra of the I–model.** In Section 3.2 we defined the deformed model with the action (3.3), which should be equivalent to the A–model of $\mathbb{P}^1$. The corresponding T–dual model is the I–model, which is a deformation of the theory discussed in the previous section. The action of this theory is given by formula (3.4), and for a more general toric variety $\mathbb{P}_S$ it is given by formula (4.4). In this section we will determine the chiral algebra of integrals of motion of this theory in the sense of [35, 12]. In this context, the I–model is analogous of the conformal A_n Toda field theory, in which the chiral algebra is the $\mathcal{W}_n$–algebra (see [8, 12]). We will show that the chiral algebra

in the I–model corresponding to a toric variety $\mathbb{P}_S$ is isomorphic to the space of global sections of the chiral de Rham complex of $\mathbb{P}_S$. In order to do this, we will identify the complex whose zeroth cohomology is this chiral algebra with the complex introduced by Borisov in [4]. The cohomology of this complex is isomorphic to the cohomology of the chiral de Rham complex of $\mathbb{P}_S$, as shown in [4]. Thus, the I–model provides a natural link between Borisov's complex, and hence the chiral de Rham complex of $\mathbb{P}_S$, and the sigma model of $\mathbb{P}_S$.

Consider first the case of $\mathbb{P}^1$. The action of the I–model given by formula (3.4) is obtained by deforming the action (2.12) of the free conformal field theory using the operators $q^{1/2}e^{iU}\pi\overline{\pi}$ and $q^{1/2}e^{-iU}\pi\overline{\pi}$. According to formula (3.5), for any chiral field $A(z)$ of the free field theory, we have in the I–model

$$(\partial_{\overline{z}}A)(z,\overline{z}) = q^{1/2}\int e^{iU(w,\overline{z})}\pi(w)\overline{\pi}(\overline{z})dw \cdot A(z) + q^{1/2}\int e^{-iU(w,\overline{z})}\pi(w)\overline{\pi}(\overline{z})dw \cdot A(z).$$

Therefore the chiral algebra of the I–model is equal to the intersection of the kernels of the operators $\int e^{\pm iU(w,\overline{z})}\pi(w)\overline{\pi}(\overline{z})dw$ on the chiral algebra V of the free theory.

The chiral algebra of the free conformal field theory is given by the direct sum

$$(5.8) \qquad\qquad V = \bigoplus_{r\in\mathbb{Z}} \widetilde{\mathcal{F}}^{\mathrm{ch}}_{r,r,0} \otimes \mathcal{F}_{\mathrm{ferm}}.$$

Here $\widetilde{\mathcal{F}}^{\mathrm{ch}}_{r,r,0}$ is the chiral sector of the Fock representation $\widetilde{\mathcal{F}}_{r,r,0}$ introduced in Section 5.2. The corresponding chiral fields are normally ordered products of $\partial_z U(z), \partial_z R(z)$ and their derivatives, as well as the fields $e^{2rR_-(z)}$, where R_- is given by formula (5.7). The chiral fermionic fields corresponding to vectors in the chiral fermionic Fock representation $\mathcal{F}_{\mathrm{ferm}}$ introduced in Section 5.1 are normally ordered products of $\psi(z), \pi(z)$ and their derivatives.

We need to find the intersection of the kernels of the operators $\int e^{\pm iU(w,\overline{z})}\pi(w)\overline{\pi}(\overline{z})dw$ on V.

Let us write $U(w,\overline{w}) = U_-(w) + U_+(\overline{w})$, where

$$U_-(w) = \frac{1}{2}(U_0 + p_U^-\log w) + \sum_{n\neq 0}\frac{U_n}{n}w^{-n},$$

$$U_+(\overline{w}) = \frac{1}{2}(U_0 + p_U^+\log\overline{w}) + \sum_{n\neq 0}\frac{\overline{U}_n}{n}\overline{w}^{-n},$$

and $p_U^\pm = p_U \pm i\widetilde{\omega}$. Then it is clear that the kernel of the operator

$$\int e^{\pm iU(w,\overline{z})}\pi(w)\overline{\pi}(\overline{z})dw = e^{\pm iU_+(\overline{z})}\overline{\pi}(\overline{z})\int e^{\pm iU_-(w)}\pi(w)dw$$

on V is equal to the kernel of the operator

$$S_\pm = \int e^{\pm iU_-(w)}\pi(w)dw.$$

This allows us to express the chiral algebra of the I–model purely in terms of modules over a free chiral superalgebra, as we now explain.

Consider the Heisenberg-Clifford superalgebra with the generators A_n, B_n, Φ_n, Ψ_n, $n \in \mathbb{Z}$, and relations

$$[B_n, A_m] = n\delta_{n,-m}, \qquad [\Phi_n, \Psi_m]_+ = \delta_{n,-m},$$

with all other super-commutators being zero. Let $F_{a,b}$ be the Fock representation of this algebra generated by a vector $|a, b\rangle$ which is annihilated by all generators with $n > 0$ and such that

$$A_0|a,b\rangle = a|a,b\rangle, \qquad B_0|a,b\rangle = b|a,b\rangle, \qquad \Psi_0|a,b\rangle = 0.$$

The direct sum $\bigoplus_{a,b\in\mathbb{Z}} F_{a,b}$ is a chiral algebra. In particular, the field corresponding to the vector $|a, 0\rangle$ is given by the standard formulas

$$e^{a\int A(z)dz} = \exp\left(ap_A + aA_0\log z - a\sum_{n\neq 0}\frac{A_n}{n}z^{-n}\right).$$

Let us identify the above chiral algebra with our chiral algebra by the formula

$$A_n \mapsto -iU_n, \qquad B_n \mapsto 2R_n, \qquad n \neq 0,$$

$$A_0 \mapsto \frac{i}{2}p_U^-, \qquad B_0 \mapsto -R_0^-,$$

$$\Phi_n \mapsto \psi_n, \qquad \Psi_n \mapsto i\pi_n, \qquad n \in \mathbb{Z}.$$

Then our chiral algebra V given by formula (5.8) becomes $F_{0,\bullet} = \bigoplus_{b\in\mathbb{Z}} F_{0,b}$, and the above operators $S_\pm$ become the operators

(5.9) $$S_\pm = -i\int e^{\pm\int A(z)dz}\Psi(z)dz: \quad F_{0,\bullet} \to F_{\pm 1,\bullet}.$$

Thus, we obtain that the chiral algebra of the I–model is the intersection of the kernels of the operators S_+ and S_- on $F_{0,\bullet}$.

Now let us compare this with the results of [4] (see also [17]). In that paper a complex $C^\bullet$ is constructed such that $C^0 = F_{0,\bullet}$, and $C^n = F_{n,\bullet} \oplus F_{-n,\bullet}$, where $F_{n,\bullet} = \bigoplus_{b\in\mathbb{Z}} F_{n,b}$. The differential is $d = S_+ + S_-$, where $S_\pm : F_{n,\bullet} \to F_{n\pm 1,\bullet}$ is given by formula (5.9), if $\pm n \geq 0$ and is equal to 0 otherwise. It is proved in [4] that the nth cohomology of this complex is isomorphic to the nth cohomology of the chiral de Rham complex of $\mathbb{P}^1$. The latter vanishes for $n \neq 0, 1$, and the 0th and 1st cohomology may be described as modules over the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ of level 0 [17] (see Remark 5.1 below).

Now we see that this complex, after a change of variables, naturally appears in the context of the I–model, and hence the A–model of $\mathbb{P}^1$, as anticipated in [4]. In particular, we find that the 0th cohomology of the chiral de Rham complex of $\mathbb{P}^1$ is isomorphic to the chiral algebra of the I–model, and hence to the chiral algebra of the A–model associated to $\mathbb{P}^1$ (in the infinite volume limit).

The operators $S_\pm$ are analogues of the screening operators familiar from the theory of $\mathcal{W}$–algebras (see [8, 12]). It is clear from the above formula that they are residues of fermionic fields. Screening operators of this type have been considered by B. Feigin [9].

The generalization of the above computation to the case of the I–model associated to a toric variety $\mathbb{P}_S$ is straightforward. Using a change of variables similar to the one explained above, we relate the chiral algebra of the I–model associated to $\mathbb{P}_S$ to the 0th

cohomology of a complex constructed in [4]. According to [4], the cohomologies of this complex are isomorphic to the cohomologies of the chiral de Rham complex of $\mathbb{P}_S$. In particular, we find that the chiral algebra of the I–model associated to $\mathbb{P}_S$ is isomorphic to the 0th cohomology of the chiral de Rham complex of $\mathbb{P}_S$.

Remark 5.1. The toric sigma model carries a chiral $\widehat{\mathfrak{sl}}_2$ symmetry with level 0, with the generating currents given by the formulas

$$J^{\pm}(z) = \,:(p(z) \pm \psi(z)\pi(z))e^{\pm X(z)}:\,, \qquad J^0(z) = -ip(z).$$

These formulas can be obtained by a change of variables from the formulas found in [10], which constitute a special case of the Wakimoto free field realization. There is also an anti-chiral copy of $\widehat{\mathfrak{sl}}_2$, with the anti-chiral currents given by similar formulas. The above chiral fields commute with the screening operators and therefore survive in the deformed theory, and hence we obtain that the A–model of $\mathbb{P}^1$ carries an $\widehat{\mathfrak{sl}}_2$ symmetry. It corresponds to the natural action of the Lie algebra $\mathfrak{sl}_2$ on $\mathbb{P}^1$. One can check that these currents are $\overline{Q}(q)$–exact, where $\overline{Q}(q)$ is the right supercharge discussed in the next section. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.4. **Cohomology of the right moving supercharge.** Now we wish to compute the cohomology of the right moving supercharge of the I–model and its degeneration. Consider the case of $\mathbb{P}^1$. Recall from Section 3.3 that the right moving supercharge of the I–model is given by the formula

$$\overline{Q}(q) = \int \left(\overline{\psi}\partial_{\overline{z}}U\,d\overline{z} + q^{1/2}\left(e^{iU} - e^{-iU}\right)\pi dz \right),$$

and in the T–dual variables by

$$\overline{Q}(q) = \int \left(\overline{\psi}\overline{p}d\overline{z} + q^{1/2}\left(e^{i\int P} - e^{-i\int P}\right)\pi dz \right)$$

(we omit the factor of $-i$ which is inessential for the computation of cohomology). We wish to compute the cohomology of this operator on the Hilbert space $\mathcal{H}$ of our theory that was described in Section 5.1. As the space of functions of the zero mode of the bosonic fields $X, \overline{X}$ we will take the space of all smooth functions on $\mathbb{C}^{\times}$.

We will compute the cohomology of $\overline{Q}(q)$ by utilizing a spectral sequence corresponding to a $\mathbb{Z}$–bigrading on $\mathcal{H}$ (we note that our computation is similar in spirit to the computation in [32]). The only non-zero degrees are assigned to the fermionic generators:

$$\deg\overline{\psi}_n = -\deg\overline{\pi}_n = (1,0), \qquad \deg\pi_n = -\deg\psi_n = (0,1).$$

Then the first summand of the differential $Q(q)$ has degree $(1,0)$, while the second summand has degree $(0,1)$. Using the additional $\mathbb{Z}$–gradings by the eigenvalues of the L_0 and $\overline{L}_0$ operators, it is easy to see that the corresponding spectral sequence converges. The zeroth differential is

$$\int \overline{\psi}\overline{p}d\overline{z} = \sum_{n\in\mathbb{Z}} \overline{\psi}_n\overline{p}_{-n}.$$

Clearly, it affects only the part of the complex which is generated by $\overline{p}_n, \overline{X}_n, \overline{\psi}_n, \overline{\pi}_n$. All non-zero modes of these operators cancel out in the cohomology, and the cohomology reduces to the cohomology of the operator $\overline{\psi}_0 \overline{p}_0$ on the zero mode part of the complex. This operator is the Dolbeault $\overline{\partial}$ operator, and its cohomology is the space of holomorphic functions on $\mathbb{C}^\times$ in degree zero, and the other cohomology vanishes. In the computation that follows we will replace this space by the space of Laurent polynomial functions on $\mathbb{C}^\times$. This will not affect the cohomologies.

Thus, we obtain that the first term of the spectral sequence is given (in the variables of the I–model) by the direct sum

$$\bigoplus_{m,r \in \mathbb{Z}} \widetilde{\mathcal{F}}^{\mathrm{ch}}_{r,r,m} \otimes \mathcal{F}_{\mathrm{ferm}},$$

where $\widetilde{\mathcal{F}}^{\mathrm{ch}}_{r,r,m}$ is the chiral sector of the Fock representation $\widetilde{\mathcal{F}}_{r,r,m}$ introduced in the previous section. The cohomological gradation corresponds to the fermionic charge operator. The differential is given by the formula

$$d = q^{1/2} \int \left(e^{iU} - e^{-iU} \right) \pi dz.$$

To relate this complex to the complex considered in [4, 25], we make the change of variables from the previous section. Then the complex becomes

$$C_q = \bigoplus_{r,m \in \mathbb{Z}} F_{m,r}$$

with the differential $d = q^{1/2} S_+ - q^{1/2} S_-$, where $S_\pm$ are the screening operators from the previous section. Thus, as a vector space, this complex coincides with the complex $C^\bullet$ from the previous section, but the differential is different. Indeed, the differential on $C^\bullet$ was given by formula $S_+ - S_-$ (up to inessential factors), but by definition S_+ acted non-trivially on $F_{m,r}$ with $m \geq 0$, and by 0 on $F_{m,r}$ with $m < 0$, whereas S_- acted non-trivially on $F_{m,r}$ with $m \leq 0$, and by 0 on $F_{m,r}$ with $m > 0$. In contrast, now the differential is defined in such a way that both S_+ and S_- act non-trivially on $F_{m,r}$ with an arbitrary integer m. In particular, the cohomological gradation on $C^\bullet$ introduced in the previous section is now well-defined only mod 2.

This new complex is therefore a deformation of the complex $C^\bullet$, which was previously considered in [4, 25]. It was shown in [25] that its cohomology is isomorphic to the *quantum cohomology* of $\mathbb{P}^1$ (so it is commutative as a chiral algebra). The cohomology is therefore two-dimensional, and as representatives of two independent cohomology classes we can take the identity operator and the operator $q^{1/2}(e^{iU} + e^{-iU})$, familiar from the Landau-Ginzburg theory.

But what about the complex $C^\bullet$ considered in the previous section? Following [4, 25], we can interpret it as a a certain limit of the complex $C^\bullet_q$ when $q \to 0$. To this end, let us redefine the term $F_{m,r}$ of the complex by multiplying it with $q^{|m|/2}$. Then the differential $q^{1/2} S_+$, when acting from $F_{m,r}$ to $F_{m+1,r}, m \geq 0$, will become S_+, but when acting from $F_{m,r}$ to $F_{m+1,r}, m < 0$, it will become qS_+, and so will vanish when $q = 0$. Likewise, $q^{1/2} S_-$, when acting from $F_{m,r}$ to $F_{m-1,r}, m \leq 0$, will become S_-, but when acting from $F_{m,r}$ to $F_{m-1,r}, m > 0$, it will become qS_-, and so will vanish when $q = 0$.

Thus, when $q = 0$ the complex C_q will degenerate into the complex $C^\bullet$ considered in the previous section. Hence its cohomology will become isomorphic to the cohomology of the chiral de Rham complex of $\mathbb{P}^1$ (and so will become much bigger).

The degenerate complex makes perfect sense as the complex computing the cohomology of the right moving supercharge in the *perturbative* regime, i.e., without the instanton corrections. Indeed, in the perturbative regime we consider maps $\Sigma \to \mathbb{P}^1$ which either pass through 0 or through ∞, but not through both points. We achieve this effect by rescaling the terms of the complex as described above. According to [34, 20], we should expect that the cohomology of the right moving supercharge of the A–model of $\mathbb{P}^1$ (which is equivalent to the I–model) is isomorphic to the cohomology of the chiral de Rham complex of $\mathbb{P}^1$. The above computation confirms this assertion. In addition, we have also obtained the cohomology of the right moving supercharge with the instanton corrections and found it to be isomorphic to the quantum cohomology of $\mathbb{P}^1$, using the results of [4, 25].

To summarize, we have a family of complexes C_q depending on a complex parameter q. When $q \neq 0$ the cohomology is two-dimensional and is isomorphic to the quantum cohomology of $\mathbb{P}^1$, and when $q = 0$ the cohomology is isomorphic to the cohomology of the chiral de Rham complex of $\mathbb{P}^1$. Note that we also have a residual action of the left moving supercharge on this cohomology. For $q \neq 0$ it simply acts by zero, and so the cohomology of the total supercharge is the quantum cohomology of $\mathbb{P}^1$ as expected in the A–model of $\mathbb{P}^1$. If $q = 0$, then it is known (see [25]) that the cohomology will be the ordinary (not quantum) cohomology of $\mathbb{P}^1$.

This pattern holds for other Fano toric varieties. Indeed, we can show in the same way as above that for such a variety $\mathbb{P}_S$ the cohomology of the right moving supercharge of the I–model introduced in Section 4.5 is computed by a complex isomorphic to the one introduced in [4, 25]. The differential obtained from the supercharges of Section 3.3 coincides with the differential of [4, 25]. In fact, we have a family of complexes parameterized by the Kähler cone of $\mathbb{P}_S$. According to [25], in the case when $\mathbb{P}_S = \mathbb{P}^n$ its cohomology is isomorphic to the quantum cohomology of $\mathbb{P}^n$. We expect the same to be true for general Fano toric varieties. This is confirmed by the computation in [25] (which uses the results of [1]) which shows that the cohomology of the total supercharge is isomorphic to the quantum cohomology of $\mathbb{P}_S$. Moreover, the cohomology classes are represented by the elements of the gradient ring of the superpotential $\widetilde{W}$ of the I–model. This is what we expect to be true in the I–model of $\mathbb{P}_S$, which should be equivalent to the A–model of $\mathbb{P}_S$.

In the limit when the parameters of our complex tend to zero, our complex degenerates. The cohomology of the degenerate complex was shown in [4] to be isomorphic to the cohomology of the chiral de Rham complex of $\mathbb{P}_S$. This is again in agreement with the assertion of [34, 20].

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